One-sided Limits

**Definition** For a function $f$ and a number $c$,

$$\lim_{x\to c^-} f(x)$$

means the limit of $f(x)$ as $x$ approaches $c$ from the left ($-$) side.

$$\lim_{x\to c^+} f(x)$$

means the limit of $f(x)$ as $x$ approaches $c$ from the right ($+$) side.

These are called **one-sided** limits. The regular limit is called the **two-sided** limit.
Find the limits.

- \( \lim_{x \to a^-} f(x) \)
- \( \lim_{x \to b^+} f(x) \)  # means “none of these”
- \( \lim_{x \to c^-} f(x) \) (A) 1 (B) 2 (C) 3 (D) 4 (E) #
- \( \lim_{x \to c^+} f(x) \) (A) 1 (B) 2 (C) 3 (D) 4 (E) #
- \( \lim_{x \to d^-} f(x) \) (A) 1 (B) 2 (C) 3 (D) 4 (E) #
- \( \lim_{x \to e^+} f(x) \) (A) 1 (B) 2 (C) 3 (D) 4 (E) #

**Fact** The two-sided limit exists iff both one-sided limits exist and have the same value. The 2-sided at \( b \) at \( c \)?
\[ \lim_{x \to 0^-} \left( \frac{5}{3 + \frac{1}{x}} \right) \]. First try calculating the two-sided limit. If the two-sided exists it is also the one-sided limit.

\[ = \lim_{x \to 0} \left( \frac{5x}{3x + 1} \right) = \frac{5 \cdot 0}{3 \cdot 0 + 1} = 0. \]

If \( \lim_{x \to a^+} \) or \( \lim_{x \to a^-} \) is nonzero/0, write \( \infty \) or \( -\infty \) instead of “d.n.e.”. To determine if it is \( \infty \) or \( -\infty \), plug in a number \( a^+ \) (or \( a^- \)) which is slightly larger (or smaller) than \( a \).

\[ \lim_{x \to 2^+} \frac{x}{x - 2} = \frac{2^+}{2^+ - 2} = \frac{+}{0^+} = +\infty \]

\[ \lim_{x \to 2^-} \frac{x}{x - 2} = \frac{2^-}{2^- - 2} = \frac{+}{0^-} = -\infty \]

\[ \lim_{x \to 2^+} \frac{3}{2 - x} = (A) -\infty \quad (B) 0 \quad (C) \infty \quad (D) \text{d.n.e.} \quad (E) \# \]

\[ \lim_{x \to 2^-} \frac{3}{2 - x} = (A) -\infty \quad (B) 0 \quad (C) \infty \quad (D) \text{d.n.e.} \quad (E) \# \]
Note the graphs of $\sin x$, $x$, $\tan x$ as $x \to 0^+$. 

As $x \to 0$ we have $1 - \cos x$ goes to 0 faster than $x$ and we have $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$.

However $\tan x$, $x$, $\sin x$ all go to 0 at the same speed. For each graph, the tangent at 0 is 1.
As \( x \to 0^+ \) we have
\[
\sin(x) \leq x \leq \tan(x) = \frac{\sin x}{\cos x}.
\]
\[\therefore 1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \quad -- \text{divide the above line by } \sin x
\]
\[\therefore \lim_{x \to 0^+} 1 \leq \lim_{x \to 0^+} \frac{x}{\sin x} \leq \lim_{x \to 0^+} \frac{1}{\cos x}.
\]
\[\therefore 1 \leq \lim_{x \to 0^+} \frac{x}{\sin x} \leq \frac{1}{1} = 1 \quad \text{and so } \lim_{x \to 0^+} \frac{\sin x}{x} = 1.
\]

The case for \( x \to 0^- \), is similar but with the inequality being \( \frac{\sin x}{\cos x} = \tan(x) < x < \sin(x) \). Since both one-sided limits are 1, so is the two-sided limit: \( \lim_{x \to 0} \frac{x}{\sin x} = 1 \).

Taking reciprocals gives \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

**Theorem**

\[\lim_{x \to 0} \frac{x}{\sin x} = 1, \quad \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0\]
**THEOREM**

\[ \lim_{x \to 0} \frac{x}{\sin x} = 1, \quad \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \]

In the theorem, \( x \) can be replaced by any variable, say \( h \), or, in fact, any term which goes to 0 as \( x \) goes to 0. Hence

\[ \lim_{x \to 0} \frac{\sin x}{x} = \lim_{y \to 0} \frac{\sin y}{y} = \lim_{x \to 0} \frac{\sin 2x}{2x}. \]

\[ \lim_{x \to 0} \frac{\tan x}{x} = ? \] Note \( \frac{\tan x}{x} = \frac{1}{\cos x} \frac{\sin x}{x} \)

\[ \lim_{x \to 0} \frac{x \sin(3x)}{\sin(2x) \sin(4x)} = \lim_{x \to 0} \frac{x}{1} \frac{\sin(3x)}{\sin(2x)} \frac{1}{\sin(4x)} \]

\[ = \lim_{x \to 0} \frac{x}{1} \frac{\sin(3x)}{3x} \frac{2x}{\sin(2x)} \frac{4x}{\sin(4x)} \frac{(3x)}{(2x)(4x)} \]

\[ = \lim_{x \to 0} \frac{\sin(3x)}{3x} \frac{2x}{\sin(2x)} \frac{4x}{\sin(4x)} \frac{3}{8} \]

\[ = (1)(1)(1)(\frac{3}{8}) = \frac{3}{8}. \]

\[ \lim_{x \to 0} \frac{\sin(4x)}{2x} = \] (A) 0  (B) 1  (C) 2  (D) 4  (E) #