Odd degree roots or vertical asymptotes: the sign changes, one side is positive and the other negative. \(-x, x^3, 1/x, -1/x\).

Even degree roots or vertical asymptotes: the sign doesn’t change, both sides are positive or both are negative, as \(x^2, 1/x^2, -1/x^2\).
**Asymptotes**

**Definition** For the graph of $y = f(x)$,

A vertical line $x = a$ is a *vertical asymptote (v.a.)* if either $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ is $-\infty$ or $\infty$. The graph approaches this asymptote as $x$ approaches $a$ for one side or the other.

$$\lim_{x \to a} f(x) = \pm \infty$$
A graph approaches its leading term as \( x \) approaches infinity. It approaches a \textit{horizontal asymptote (h.a.)} \( y = c \) if the infinite limit is a finite number \( c \) i.e., if \( \lim_{x \to \infty} f(x) = c \) or \( \lim_{x \to -\infty} f(x) = c \).

A graph will not cross a vertical asymptote but it may cross a horizontal one as in the picture above.

\textbf{Fact} For rational functions (ratios of two polynomials), \( \lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) \) if either one is finite. This fails for nonrational functions such as \( e^x \).
a is a *key number* if $f(a)$ is zero (an $x$-intercept) or undefined (a vertical asymptote). On the key intervals before, between or after key numbers $f(x)$ is either positive or negative. The leading term determines the sign of the first and last key interval. The *degree* of a key number is the degree (power) of its factor. For the remaining intervals, the function’s *sign changes* as you go past a key number of *odd degree*. The sign doesn’t change if it has even degree.
Draw the graph and the asymptotes labeled with their equations \((y = c \text{ or } x = a)\). Also label the zeros (roots).

\[ f(x) = \frac{2x}{x - 3} \]

\(y\)-intercept: \(y = 0\). Key numbers: \(x = 0, 3\)

\(x\)-intercept: \(x = 0\) degree 1

v.a.: \(x = 3\) degree 1

h.a.: \(\lim_{x \to \infty} \frac{2x}{x - 3} = \lim_{x \to \infty} \frac{2x}{x} = \lim_{x \to \infty} 2 = 2\). \(y = 2\) h.a.

\[ \lim_{x \to 3^-} \frac{2x}{x - 3} = -\infty \quad \lim_{x \to 3^+} \frac{2x}{x - 3} = +\infty \]
\[ y = \frac{2x - 6}{2x^3 - 8x^2}. \]

Reduce and factor:
\[ = \frac{2(x - 3)}{2(x^3 - 4x^2)} = \frac{(x - 3)}{x^3 - 4x^2} = \frac{x - 3}{x^2(x - 4)} \]

\[ \text{y-intercept: none. key numbers: } x = 0, 3, 4 \]

\[ \text{x-intercept: } x=3 \text{ degree 1} \]

\[ \text{v.a.: } x=0 \text{ degree 2, } x=4 \text{ degree 1} \]

\[ \text{lead. term: } \frac{x - 3}{x^2(x - 4)} \rightarrow \frac{x}{x^2x} = \frac{1}{x^2} \]

\[ \text{h.a.: } \lim_{x \to \pm \infty} \frac{x - 3}{x^2(x - 4)} = \lim_{x \to \pm \infty} \frac{1}{x^2} = \frac{1}{\pm \infty} = 0 \quad y = 0 \text{ h.a.} \]

As \( x \) goes to \( \pm \infty \), the leading term \( \frac{1}{x^2} \) is a positive number which approaches 0. Hence the first and last key intervals are positive. The sign changes at \( x = 3, 4 \) (they have odd degree) and doesn’t change sign at \( x = 0 \) (it has even degree). The signs of the intervals are

<table>
<thead>
<tr>
<th>Sign变化</th>
<th>x值范围</th>
<th>度数</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>0, deg 2</td>
<td></td>
</tr>
<tr>
<td>+</td>
<td>3, deg 1</td>
<td></td>
</tr>
<tr>
<td>-</td>
<td>4, deg 1</td>
<td></td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0^-} \frac{x - 3}{x^2(x - 4)} = +\infty \quad \lim_{x \to 0^+} \frac{x - 3}{x^2(x - 4)} = +\infty \]

\[ \lim_{x \to 4^-} \frac{x - 3}{x^2(x - 4)} = -\infty \quad \lim_{x \to 4^+} \frac{x - 3}{x^2(x - 4)} = +\infty \]
\[
y = \frac{2x^3 - 8x^2}{2x - 6}
\]
Reduce and factor: \[
\frac{x^2(x - 4)}{x - 3}
\]

- **y-intercept:** 0
  - (A) 0  (B) 3  (C) 4  (D) none  (E) #

- **x-intercepts:**
  - (A) 3,4  (B) 0,3,4  (C) 0,3  (D) 0,4  (E) #

- **Vertical asymptote:**
  - (A) \(x = 0\)  (B) \(x = 3\)  (C) \(x = 4\)  (D) none  (E) #

\[
\lim_{x \to \pm \infty} \frac{x^2(x - 4)}{x - 3} = \lim_{x \to \pm \infty} \frac{x^2x}{x} = \lim_{x \to \pm \infty} x^2 = (\pm \infty)^2 = \infty
\]

- **Horizontal asymptote:**
  - (A) \(y = 0\)  (B) \(y = 3\)  (C) \(y = 4\)  (D) none  (E) #

The first and last key intervals are positive. The sign doesn’t change at 0 (it has even degree) but does change at 3 and 4 (which have odd degree).
Reduce and factor: \( \frac{x^2(x - 4)}{x - 3} \)

What is the pattern of signs?

(A) \[ + + - \]

0, deg 2 3, deg 1 4, deg 1

(B) \[ + + - + \]

0 3 4

(C) \[ + - + + \]

0 3 4

(D) \[ + - - + \]

0 3 4

(E) \[ # \]
Graph: Draw the asymptotes, labeled with their equations.

x = 3
v.a.