Math 241 Lecture 9

$$f(x) = \begin{cases} 
\frac{x^2 + 4}{x - 2} & \text{if } x < 2 \\
5 & \text{if } x = 2 \\
x^3 & \text{if } x > 2 
\end{cases}$$

Is it continuous at $x = 2$?

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2 + 4}{x - 2} = 2^2 + 4 = 8$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^3 = \infty$$

$$\lim_{x \to 1} f(x) = 5.$$  

$\therefore f$ has a removable discontinuity at 2.

Notation: $\exists \quad \forall \quad \Rightarrow \quad \Leftrightarrow \quad \therefore$ mean “there exists”, “for all”, “implies”, “iff”, “therefore”.

Graph intersection, intermediate values

**Graph Intersection Theorem** If $f$ and $g$ are continuous and if $f(x) \leq g(x)$ for some $x = a$ and if $f(x) \geq g(x)$ for some $x = b$ then the graphs intersect, $f(x) = g(x)$ for some $x$ between $a$ and $b$.

![Graph Intersection Theorem](image)

**Fixed Point Theorem** If $f$ is continuous with domain $[a, b]$ and range in $[a, b]$ then there is an $x \in [a, b]$ with $f(x) = x$, i.e., $x$ is a fixed point of $f$.

**Proof.** Consider the graphs of $x$ and $f(x)$.

Since the range is in $[a, b]$, $a \leq f(x) \leq b$ for $x \in [a, b]$.

$\therefore a \leq f(a)$ and $b \geq f(b)$.

By the Graph Intersection Theorem, the graphs of $x$ and $f(x)$ cross: $x = f(x)$ for some $x \in [a, b]$.  \[\square\]

A stretched rubber band lies on the $x$-axis stretched across the interval $[-1, 2]$ when it is released, it lies in the interval $[0, 1]$. Use the Graph Intersection Theorem to prove that some point of the rubber band doesn’t move. For a point originally at position $x \in [-1, 2]$ on the stretched rubber band, let $f(x) \in [0, 1]$ be its position after being released. We must prove that $f(x) = x$ for some $x \in [-1, 2]$.

**Proof.**

This is easy using the Fixed Point Theorem but we are to use the Graph Intersection Theorem.

When $x = -1$, $f(x) = f(-1) = 0$, hence $x \leq f(x)$.

When $x = 2$, $f(x) = f(2) = 1$, hence $x \geq f(x)$.

By the Graph Intersection Theorem, the graphs of $x$ and $f(x)$ cross: $x = f(x)$ for some $x \in [-1, 2]$.  \[\square\]

**Intermediate Value Theorem** Suppose $f$ is continuous. If $C$ is between two values $f(a)$ and $f(b)$, then $C = f(x)$ for some $x$ between $a$ and $b$.

**Proof.** Consider the graphs of $f(x)$ and of the constant function $C$.

When $x = a$, $f(x) = f(a) \leq C$. Hence $f(x) \leq C$.

When $x = b$, $f(x) = f(b) \geq C$. Hence $f(x) \geq C$.

By the Graph Intersection Theorem, the graphs of $C$ and $f(x)$ cross: $C = f(x)$ for some $x$ between $a$ and $b$.  \[\square\]

- Does $x^4 + x^2 + 1 = 0$ have a solution?

- Does $x^3 + x + 1 = 0$ have a solution?

Let $f(x) = x^3 + x + 1$. Then $f$ has values above and below 0.

$f(-1) = -1 + -1 + 1 = -1$

$f(1) = 1 + 1 + 1 = 3$

0 is between $f(-1) = -1$ and $f(1) = 3$.

Hence there is an $x$ between -1 and 1 such that $f(x) = 0$, i.e., $x^3 + x + 1 = 0$.  \[\square\]