Derivatives and rates of change

For any number $c$, the derivative of $f$ at $c$

$= f'(c)$

$= \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$

$= \text{the slope of } y = f(x) \text{ at } c$

$= \text{the slope of the tangent to } y = f(x) \text{ at } x = c$

$= \text{the rate of increase of } f \text{ at } c$

$= \text{the rate of change of } f \text{ at } c$

$= \text{the rate of change of } f \text{ with respect to } x \text{ at } x = c.$
For a function $f(x)$. The rate of decrease at $c$ is

(A) $f'(x)$
(B) $f'(c)$
(C) $-f'(x)$
(D) $-f'(c)$
(E) # (none of these)
“/” and the word “per” often indicate a rate of change.
miles/hour = the rate of change of distance w.r.t. time.
dollars/lb. = the rate the price changes w.r.t. the weight.
cubic inches/sec. = the rate a volume changes w.r.t. time.

If $d(t)$ = the distance in miles traveled at time $t$ hours then $d'(t) = \frac{d}{dt}$ = the rate of change of distance w.r.t. time $t$ = the speed in miles/hour.

If $p(w) = \text{the price of a bag of } w \text{ pounds of flour in dollars,}$ then $p'(w) = \frac{dp}{dw}$ = the rate the bag’s price changes w.r.t. its weight $w$ = its price in dollars per pound.

If $V(t)$ is the volume of a balloon at time $t$, then $V'(t) = \frac{dV}{dt}$ = the rate the volume changes w.r.t. time $t$ = the rate of the volume increases per unit of time.

If volume is in cubic inches and time in seconds, the rate of change in volume will be in cubic inches per second.
A stone is dropped from a 500 foot bridge. Its position at time $t$ seconds is $p(t) = 500 - 8t^2$ feet. How fast is it falling one second after the drop?

**Ans:** The derivative is $p'(t) = (500 - 8t^2)' = -16t$.
The speed at time one second is $p'(1) = -16$ feet/second.

For the same stone with position $p(t) = 500 - 8t^2$ and derivative $p'(t) = -16t$ what is its speed 2 seconds after the drop?

(A) -16 feet/second
(B) -24 feet/second
(C) -32 feet/second
(D) -160 feet/second
(E) # (none of these)
The surface area of an oil spill at time $t$ days is $A(t) = 10t^2$ square miles. How fast is the area of the spill increasing?

(A) $10t$  (B) $20t$  (C) $30t$  (D) $40t$  (E) $50t$

For the same oil spill, how fast is the area of the spill increasing at time two days?

(A) 10  (B) 20  (C) 30  (D) 40  (E) 50

What are the units?

(A) miles  (B) days  (C) miles/day  (D) days/mile  (E) square miles/day
Double-D notation

Sometimes when writing a function we may use a dependent variable $y$ for $f(x)$ and write

$$y = x^2 \text{ instead of } f(x) = x^2.$$ 

In this case we will write

$$y' \text{ or } \frac{dy}{dx} \text{ instead of } f'(x),$$

and write

$$\frac{dy}{dx} \bigg|_{x=2} \text{ instead of } f'(2)$$

The following are various notations for the derivative.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f$$
For \( f(x) = \cos x \), \( f'(x) = -\sin(x) \).

If \( y = \cos x \), then \( \frac{dy}{dx} = -\sin x \) and
\[
\frac{dy}{dx} \bigg|_{x=\pi/2} = -\sin(\pi/2) = -1.
\]

For \( f(x) = \sin x \), \( f'(x) = \cos(x) \).

If \( y = \sin x \), then \( \frac{dy}{dx} = 
\[
(A) \ \sin x \quad (B) -\sin x \quad (C) \ \cos x \quad (D) -\cos x \quad (E) \#
\]
\[
\frac{dy}{dx} \bigg|_{x=0} = 
\[
(A) 0 \quad (B) 1 \quad (C) -1 \quad (D) \pi/2 \quad (E) \#
\]