Math 241  Lecture 15

Trigonometric derivatives

**Trigonometric identities**

\[
\sin^2 x + \cos^2 x = 1
\]
\[
\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}
\]
\[
\sec x = \frac{1}{\cos x} \quad \csc x = \frac{1}{\sin x}
\]

**Trigonometric derivatives**

\[
\sin' x = \cos x \quad \cos' x = -\sin x
\]
\[
\tan' x = \sec^2 x \quad \cot' x = -\csc^2 x
\]
\[
\sec' x = \sec x \tan x \quad \csc' x = -\csc x \cot x
\]

Dual functions starting with "c" have derivatives starting with "-".

- \( \tan' x = \frac{\sin x}{(\cos x)^2} \)
- \( \sec' x = \frac{1}{\cos x} \)

**Chain rule**

Write the following as a composition \( f(g(x)) \) of an outer function \( f \) and an inner function \( g \).

- \((x^2 + 1)^6 \) Start with \( x \) and work inside out.
  - The inner function acts on \( x \) first.
  - The outer function then does what remains to be done.
  - You can get this by replacing the inner function with \( x \).

  - inner function: \( g(x) = x^2 + 1 \) or \( g(x) = x^2 \)
  - outer function: \( f(x) = x^6 \) or \( f(x) = (x + 1)^6 \)

- \( \sqrt{1-x^2} \)
  - inner function: \( g(x) = \)
    - (A) \( \sqrt{x} \)
    - (B) \( 1 - x \)
    - (C) \( x^2 \)
    - (D) \( 1 - x^2 \)
    - (E) \( \sqrt{1-x} \)
  - outer function: \( f(x) = \)
    - (A) \( \sqrt{x} \)
    - (B) \( 1 - x \)
    - (C) \( x^2 \)
    - (D) \( 1 - x^2 \)
    - (E) \( \sqrt{1-x} \)

**Chain rule** If \( f \) and \( g \) are differentiable, then so is \( f(g(x)) \) and

\[
(f(g(x)))' = f'(g(x)) \cdot g'(x)
\]

- \( (x^2 + 1)^6 \)' = ?

**Detailed way.** \((x^2 + 1)^6 = f(g(x))\) where

- \( g(x) = x^2 + 1 \), \( f(x) = x^6 \)
- \( g'(x) = 2x \), \( f'(x) = 6x^5 \)

\[
((x^2 + 1)^6)' = (f(g(x)))' = f'(g(x))g'(x) = 6(x^6)(2x) = 12x(x^2 + 1)^5
\]

**Direct way.**

\[
((x^2 + 1)^6)' = 6(x^2 + 1)^5(2x) = 12x(x^2 + 1)^5
\]

Recall: \( (\sqrt{x})' = \frac{1}{2\sqrt{x}} \) \( (\frac{1}{x})' = -\frac{1}{x^2} \)

- \( \sqrt{\tan x} ' = \frac{1}{2\sqrt{\tan x}} \sec^2 x \)
- \( \tan(\sqrt{x}) ' = \)
  - (A) \( \sqrt{x} \)
  - (B) \( \frac{1}{\sqrt{x}} \)
  - (C) \( \frac{1}{2\sqrt{x}} \)
  - (D) \( \sec^2(\sqrt{x}) \)
  - (E) \( \frac{1}{2\sqrt{x}} \)

- \( (\sqrt{1-x^2})' = ? \)
  - (A) \( \frac{1}{2\sqrt{2x}} \)
  - (B) \( \frac{-2x}{\sqrt{1-x^2}} \)
  - (C) \( \frac{1}{2\sqrt{1-x^2}} \)
  - (D) \( \frac{-x}{\sqrt{1-x^2}} \)

**The chain rule in \( \frac{dy}{dx} \)-notation.**

Suppose \( y = f(u) \) and \( u = g(x) \), then \( y = f(g(x)) \) and

\[
\frac{dy}{dx} = f'(u) \quad \frac{du}{dx} = g'(x)
\]

\[
\frac{dy}{dx} = (f(g(x)))' = f'(g(x)) \cdot g'(x)
\]

\[
= f'(u) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}
\]

**Theorem** If \( y \) is a function of \( u \) and \( u \) is a function of \( x \) then

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

- \( y = u^5 \) and \( u = x^2 + 5x \), \( \frac{dy}{dx} = \)

One way

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} (x^2 + 5x)^5 = 5(x^2 + 5x)^4(2x + 5)
\]

**Better way**

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4(2x + 5) = 5(x^2 + 5x)^4(2x + 5)
\]

**Final answer for \( \frac{dy}{dx} \) must be in \( x \) and no other variable.**

For \( \frac{dw}{dz} \), the answer would have to be in \( z \).
For \( \frac{dy}{dt} \) the answer would be in \( t \) and no other variable.

- \( y = \frac{1}{u}, \quad u = x^2 - 1, \quad x = t + 3, \quad \frac{dy}{dt} = \) unsimplified answer

\[
\frac{dy}{dt} = \frac{dy}{du} \frac{du}{dx} \frac{dx}{dt} = \frac{-1}{u^2}(2x)(1) = \text{ careful}
\]

(A) \( \frac{-1}{u^2}(2x)(1) \)
(B) \( \frac{-1}{(x^2 - 1)^2}(2x)(1) \)
(C) \( \frac{-1}{u^2}[2(t + 3)](1) \)
(D) \( \frac{-1}{(x^2 - 1)^2}[2(t + 3)](1) \)
(E) #