Math 241  Lecture 18

Rate of change problems

Today’s problems involve rates of change w.r.t. time \( t \) and “Rate of change” will mean “rate of change w.r.t. \( t \)”. They will have two or more variables related by an equation. We want the rate of change of one variable given the rate of change of the other(s).

To get full credit, show the following steps:

**Picture & variables** Name the quantities involved with variables. Indicate the variables on a picture if possible.

**Want** State the wanted rate of change.

**Given** State the given facts; the known rate of change and equations in the variables. The equations may be stated explicitly in the problem or be derived implicitly via the Pythagorean Theorem or an equality between ratios of corresponding sides of similar triangles. Here, you just state the facts without simplification.

**Eq.** Write an equation relating the variable with the wanted rate of change with the variable(s) with known rates of change and no other variables. It might be a given.

**Diff.** Differentiate this equation w.r.t. \( t \). Solve for unknown rate of change.

**Ans.** Substitute in the instantaneous values. Include the units if any. The solution may not use any new variables you’ve introduced. Do this substitution after the differentiation step.

Recall: If \( y \) is a function of time \( t \), then the rate of change = rate of increase of \( y = \frac{dy}{dt} \), the rate of decrease of \( y = -\frac{dy}{dt} \)

Do not misinterpret statements like “find the rate of change when \( y = 10 \)” as implying that \( y \) is a constant or that \( y = 10 \) is a given fact. “when \( y = 10 \)” means “at the instant that \( y = 10 \)”, not that \( y \) is always 10.

- \( f(x) = x^2, f'(x) = 2x. \)

(A) 2x  (B) 20  (C) 10  (D) 0  (E) # none of these

\[ f'(10) = \]

\[ (f(10))' = \]

\[ y = x^2, \quad \frac{dy}{dx} \bigg|_{x=10} = \]

\[ x = 10, \quad y = x^2, \quad \frac{dy}{dx} = \]

Air is blown into a balloon at 4 cubic ft/min. Find the rate of change of the radius when the volume is \( 36\pi \) cubic feet. \( V = \frac{4}{3}\pi r^3, \frac{dV}{dr} = 4\pi r^2 = S = \) surface area

\[ \frac{dr}{dt}\bigg|_{V=36\pi} \]

Want

\[ \frac{dr}{dt} \]

Given \( V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = 4 \) (note, \( V = 36\pi \) is not a given)

Have \( \frac{dV}{dt} \), want \( \frac{dr}{dt} \). Need an equation relating \( V, r \).

**Eq.** \( V = \frac{4}{3}\pi r^3 \)

**Diff.** \( \frac{dV}{dt} = \frac{4\pi}{3}3r^2 \frac{dr}{dt} \)

\[ 4 = \frac{4\pi}{3}r^2 \frac{dr}{dt} \]

\[ \frac{dr}{dt} = \frac{1}{\pi r^2} \]

**Ans.** \( \frac{dr}{dt}\bigg|_{V=36\pi} = \frac{1}{\pi r^2} \quad \Rightarrow \quad r=3 \)

\[ V = 36\pi \Rightarrow 4\pi r^3/3 = 36\pi \Rightarrow r^3 = 9 \Rightarrow r = 3 \Rightarrow r = 3 \]

\[ \therefore \frac{dr}{dt}\bigg|_{V=36\pi} = \frac{1}{\pi r^2} \quad \Rightarrow \quad r = 3 \Rightarrow \frac{1}{\pi(3)^2} = \frac{1}{9\pi} \]

The radius is changing at \( 1/9\pi \) ft/min.

- A rope tied to the bow of a boat slides through a pulley on top of a 6 foot post which is on a dock at water level. If the rope is sliding through the pulley at 8 ft/min., how fast is the boat drifting away from the dock when the bow is 10 feet from the pulley (not from the dock)?

**Picture & variables**

\[ 8 \text{ft/min.} \]

Dock

\[ 6 \]

Pulley

\[ r \]

\[ x \]

\[ r = \text{radius at time } t \]

\[ \text{Vol. = } V = \frac{4}{3}\pi r^3, \quad V = 36\pi \text{ cubic feet.} \]

\[ \frac{dV}{dt} = 4\pi r^2 = S = \text{surface area} \]

\[ V = \frac{4}{3}\pi r^3 \]

**Want**

\[ \frac{dr}{dt}\bigg|_{V=36\pi} \]

**Given** \( V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = 4 \) (note, \( V = 36\pi \) is not a given)

\[ \frac{dr}{dt}\bigg|_{V=36\pi} \]

\[ \frac{dr}{dt} \quad \Rightarrow \quad r = 3 \]

\[ \frac{1}{\pi r^2} \quad \Rightarrow \quad r = 3 \]

\[ \frac{1}{9\pi} \]

The radius is changing at \( 1/9\pi \) ft/min.

\[ \frac{dr}{dt} \]

\[ \frac{dV}{dt} = 4\pi r^2 \]

\[ \frac{dV}{dt} = x \frac{dy}{dt} \]

\[ \frac{dV}{dt} = x \frac{dy}{dt} + y \frac{dz}{dt} \]