Math 241     Lecture 20

Differential approximation - linearization

For a function \( f(x) \) and a point \((a, f(a))\) on the graph, the tangent line \( y = f(a) + f'(a)(x-a) \) is the straight line which best approximates \( f \) for points \( x \) near \( a \). Solving for \( y \) gives \( y = f(a) + f'(a)(x-a) \). \( f(a) + f'(a)(x-a) \) is the \textit{linear approximation} or \textit{linearization} of \( f \) at \( x = a \). For \( x \) sufficiently near \( a \), \( f(x) \approx f(a) + f'(a)(x-a) \).

- **Linearly approximate** (tangent line) of \( f(x) = x^k \) at \( a = 1 \).
  \[
  f(x) = x^k, \quad f'(x) = kx^{k-1} \quad \text{and} \quad a = 1, \\
  f(a) + f'(a)(x-a) \\
  = f(1) + f'(1)(x-1) \\
  = 1 + k(x-1) \\
  = 1 + k \times \text{error} \\
  \text{Answer: } x^k \approx 1 + k \times \text{error} \quad \text{for } x \text{ near } 1.
  
- Approximate the value of \( .99^{13} \).
  By the above, \( x^{13} \approx 1 + 13 \times (x-1) \) for \( x \) near 1. Hence \( .99^{13} \approx 1 + 13 \times (.99 - 1) = 1 + 13 \times (-.01) = 1 - 1.3 = .87 \)

Recall, \((\ln x)' = 1/x. \quad \ln(1) = 0.

\[ \ln(x) \approx f(a) + f'(a)(x-a) = \ln(a) + \frac{1}{a}(x-a) \]

- Linearly approximate \( \ln x \) at \( a = 1 \). \( \ln(x) \approx (A) x + 1 \quad (B) x - 1 \quad (C) 1 - x \quad (D) -x \quad (E) \# \)

- Approximate the value of \( \ln(1.03) \).
  \( (A) .03 \quad (B) .97 \quad (C) .99 \quad (E) 1.01 \quad (E) 1.03 \)

Let \((x,y) = (x, f(x)) \) be a point on the \( f \)'s graph.  
Let \( dx = \Delta x = x - a \) be the change in \( x \) or \text{error in } x. 
Let \( \Delta y = y - b = f(x) - f(a) \) be the change in \( f \) or \text{error in the value of } f. 
Let \( dy = df = y - b \) be the corresponding change in \( y \) on the tangent line. Since \( f'(a) \) is the slope of the tangent line, \( \frac{dy}{dx} = f'(a) \). Solve for \( dx \) to get the \textit{differential form}:
\[
\frac{dy}{dx} = f'(a) \quad \text{or} \quad df = f'(a)dx
\]
The tangent line approximates \( f \) for points near \( a \). Hence the change \( dy \) in tangent line values approximates the change \( \Delta y \) in the function values. Hence \( \Delta y \approx dy = f'(a)dx \)

- A cylinder of height 6 inches and radius \( r \) has volume \( V = \pi r^2 h = \pi r^2 6 = 6\pi r^2 \).
  Estimate the change \( \Delta V \) in volume when the radius increases by \( dr \) inches.
  \[
  \frac{dV}{dr} = \frac{d}{dr} 6\pi r^2 = 12\pi r
  \\
  \Delta V \approx dV = 12\pi r \cdot dr \text{ inches}^3
  \\
  \text{Stated another way, suppose a hollow tube (no top or bottom) has height } h \text{ inches and radius } r \text{ and wall thickness } dr. \text{ Estimate the volume } \Delta V \text{ of this tube (just the shell } \Delta V, \text{ not the volume } V \text{ of the can).}
  \\
  V = \pi r^2 h, \quad \frac{dV}{dr} = 2\pi rh, \quad dV = \pi rh \cdot \pi \cdot \text{dr} \quad \text{inches}^3
  \\
  \Delta V = \pi \cdot \text{dr} \quad \text{inches}^3
  \\
  \text{If } y = f(x) \text{ and } x \text{ is the measured value of a variable whose actual value is } a, \text{ then } \Delta x = x - a \text{ is the measurement error. This produces an error } \Delta y = f(x) - f(a) \text{ in } y. \text{ dy will approximate this error } \Delta y. \text{ } \Delta y \approx dy = f'(a)dx
  \\
  f(x) = x^{13}, \text{ Suppose } 1 \text{ is the actual value, } x \text{ is the measured value and } dx = -.01 \text{ is the measurement error. Then } \Delta f = f(x) - f(1) \text{ is the error in the values of } f. \text{ Approximate } \Delta f \text{ with } df.
  \\
  \Delta f \approx df = f'(1)dx = 13(1)^{12} \cdot dx = 13(-.01) = -.13 \]