Differential approximation - linearization

For a function $f(x)$ and a point $(a, f(a))$ on the graph, the tangent line $y - f(a) = f'(a)(x - a)$ is the straight line which best approximates $f$ for points $x$ near $a$. Solving for $y$ gives $y = f(a) + f'(a)(x - a)$. $f(a) + f'(a)(x - a)$ is the linear approximation or linearization of $f$ at $x = a$. For $x$ sufficiently near $a$, $f(x) \approx f(a) + f'(a)(x - a)$. 

\[ f(x) = f(a) + f'(a)(x - a) \]
Linearly approximate (tangent line) of \( f(x) = x^k \) at \( a = 1 \).

\[
f(x) = x^k, \quad f'(x) = kx^{k-1} \text{ and } a = 1,
\]

\[
f(a) + f'(a)(x - a) = f(1) + f'(1)(x - 1)
\]

\[
= 1^k + k(1^{k-1})(x - 1)
\]

\[
= 1 + k(x - 1)
\]

\[
= 1 + kx - k
\]

Answer: \( x^k \approx 1 + k(x - 1) \) for \( x \) near 1.

Approximate the value of \( .99^{13} \).

By the above, \( x^{13} \approx 1 + 13(x - 1) \) for \( x \) near 1. Hence \( .99^{13} \approx 1 + 13(.99 - 1) = 1 + 13(-.01) = 1 - .13 = .87 \)
Recall, \((\ln x)' = 1/x\). \(\ln(1) = 0\)

\[
\ln(x) \approx f(a) + f'(a)(x - a) = \ln(a) + \frac{1}{a}(x - a)
\]

- Linearly approximate \(\ln x\) at \(a = 1\). \(\ln(x) \approx\)
  (A) \(x + 1\)  (B) \(x - 1\)  (C) \(1 - x\)  (D) \(-x\)  (E) #

- Approximate the value of \(\ln(1.03)\).
  (A) \(0.03\)  (B) \(0.97\)  (C) \(0.99\)  (E) \(1.01\)  (E) \(1.03\)
Let \((x, y) = (x, f(x))\) be a point on the \(f\)'s graph.

Let \(dx = \Delta x = x - a\) be the change in \(x\) or error in \(x\).

Let \(\Delta f = \Delta y = y - b = f(x) - f(a)\) be the change in \(f\) or error in the value of \(f\).

Let \(dy = df = y - b\) be the corresponding change in \(y\) on the tangent line. Since \(f'(a)\) is the slope of the tangent line, 
\[
\frac{dy}{dx} = f'(a).
\]
Solve for \(dx\) to get the differential form:

\[
dy = f'(a)dx \quad \text{or} \quad df = f'(a)dx
\]

The tangent line approximates \(f\) for points near \(a\). Hence the change \(dy\) in tangent line values approximates the change \(\Delta y\) in the function values. Hence

\[
\Delta y \approx dy = f'(a)dx
\]
A cylinder of height 6 inches and radius \( r \) has volume
\[ V = \pi r^2 h = \pi r^2 6 = 6\pi r^2. \]
Estimate the change \( \Delta V \) in volume when the radius increases by \( dr \) inches.
\[
\frac{dV}{dr} = 12\pi r \\
\frac{dV}{dr} = \frac{d}{dr} 6\pi r^2 = 12\pi r \\
\Delta V \approx dV = 12\pi r \, dr \text{ inches}^3
\]

Stated another way, suppose a hollow tube (no top or bottom) has height \( h \) inches and radius \( r \) and wall thickness \( dr \).
Estimate the volume \( \Delta V \) of this tube (just the shell \( \Delta V \), not the volume \( V \) of the can).
\[
V = \pi r^2 h, \quad \frac{dV}{dr} = 2\pi rh, \quad dV =? \\
(A) \, 2\pi rh \quad (B) \, 2\pi rh \, dr \quad (C) \, 2\pi rh \, dh \quad (D) \, \pi r^2 \, dh \quad (E) \, \# \\
\]
Estimate the volume \( \Delta V \) (in cubic inches) of the tube if the thickness is \( dr = .1 \) inches. \( \Delta V =? \)
\[
(A) \, 2\pi rh \quad (B) \, .2\pi rh \quad (C) \, .2\pi h \, dh \quad (D) \, \pi (.1)^2 \, dh \quad (E) \, \# \\
\]
If \( y = f(x) \) and \( x \) is the measured value of a variable whose actual value is \( a \), then \( \Delta x = x - a \) is the measurement error. This produces an error \( \Delta y = f(x) - f(a) \) in \( y \). \( dy \) will approximate this error \( \Delta y \). \( \Delta y \approx dy = f'(a)dx \)

\( f(x) = x^{13} \). Suppose 1 is the actual value, \( x \) is the measured value and \( dx = -0.01 \) is the measurement error. Then \( \Delta f = f(x) - f(1) \) is the error in the values of \( f \). Approximate \( \Delta f \) with \( df \).

\[
\Delta f \approx df = f'(1)dx = 13(1)^{12}dx = 13(-0.01) = -0.13
\]