Math 241 Lecture 23  Exam 2 Monday
Extrinsics: max, mins, and critical points

**Definition.** For a function \( f \) and number \( c \), \( f \) has an absolute maximum at \( c \) iff \( f(c) \geq f(x) \) \( \forall x \) in \( f \)’s domain, absolute minimum at \( c \) iff \( f(c) \leq f(x) \) \( \forall x \) in \( f \)’s domain, local maximum at \( c \) iff \( f(c) \geq f(x) \) \( \forall x \) close enough to \( c \), local minimum at \( c \) iff \( f(c) \leq f(x) \) \( \forall x \) sufficiently close to \( c \) on both sides (hence \( c \) is not an endpoint).

If \( f \) is defined on an interval \( I \) with endpoint \( c \), \( f \) has an endpoint maximum \( c \) iff \( f(c) \geq f(x) \) \( \forall x \in I \) close enough to \( c \), endpoint minimum \( c \) iff \( f(c) \leq f(x) \) \( \forall x \in I \) sufficiently close to \( c \). Points of the above types are called extremes.

- Classify the extremes.

\[ \begin{align*}
\text{(A)} & \text{ loc. max} & \text{(B)} & \text{loc. min} & \text{(C)} & \text{end. max} & \text{(D)} & \text{end. min} & \text{(E)} & \text{#} \\
\text{Classify point } e. & & & & & & & & & \\
\text{Classify point } f. & & & & & & & & & \\
\end{align*} \]

**Definition** \( c \) is a critical point of \( f \) iff

- either \( f'(c) = 0 \)
- or \( f'(c) \) undefined and \( f(c) \) is defined
- or \( c \) is an endpoint of the domain of \( f \).

- Mark the points on the \( x \)-axis which are critical points.

Remember: on critical points, \( f \) must be defined.

\[ \begin{align*}
\text{(A)} & \text{ critical point} & \text{(B)} & \text{not a critical point} \\
\text{Classify point } a. & & \text{Classify point } b. & \text{Classify } c. & \\
\end{align*} \]

Critical points are not always maximas or minimas, but maximas and minimas are always critical points.

**Critical Point Theorem** If \( f \) has an extreme at \( c \), then \( c \) is a critical point of \( f \).

To find all extremes,
- find all critical points and their values,
- classify as abs./loc./endpt. max./min. or crit. pt.

Choose the most specific: absolute max is more specific than local max which is more specific than critical point.

Write a critical point with its value: \( f(c) = b \).

For infinite intervals, find the lead term for \( f''(x) \) and determine the signs of \( f''(x) \) as usual: the sign changes when passing critical points of odd degree, doesn’t change passing critical points of even degree. \( f \) is increasing/decreasing where \( f' \) is +/-.

- List the intervals of increase and decrease and classify the critical points.

\[ f(x) = \frac{x^2 + 1}{x^2 - 1} = \frac{x^2 + 1}{(x + 1)(x - 1)} \]

write in factored form.

\[ f'(x) = \frac{-4x}{(x^2 - 1)^2} = \frac{-4(x)}{(x + 1)^2(x - 1)^2} \]

\[ f(x) = 0 \iff \mathbf{x} \iff \mathbf{x} \text{ means “never”, “none”} \]

\( f(x) \) undefined \( x = 1, -1 \)

\( f'(x) = 0 \iff x = 0 \)

\( f'(x) \) undefined \( x = 1, -1 \) not crit. pts: \( f \) is not defined on them.

lead term for \( f'(x) \): \(-4x/x^4 = -4/x^3 \)

endpts: \( \mathbf{x} \)

crit. pts: \( x = 0 \). Degree is 1.

values: \( f(0) = -1 \)

intervals: mark the undefined and crit. pts.

\( f'(x) = \frac{(-4)(x)}{(x^2 - 1)^2} \)

\[ (\mathbf{--})(\mathbf{++}) \quad (\mathbf{--})(\mathbf{+}) \quad (\mathbf{++})(\mathbf{+}) \quad (\mathbf{+})(\mathbf{+}) \quad (\mathbf{+})(\mathbf{+}) \]

Answer: \( (-\infty, -1) \quad (-1, 0) \quad [0, 1) \quad (1, \infty) \)

Answer: \( f(0) = -1 \) loc. max.

List intervals and extremes in order, left to right.

Then classify critical points as shown.

For closed (i.e., endpoints included) finite intervals, plot the values of the critical points (include the endpoints). No lead term, no pattern of signs.

- List the intervals of \( \nearrow, \searrow \) and classify critical points.

\[ f(x) = (1 - x)^{1/3} = (1 - x)^{1/3} \text{ for } x \in [-1, 1] \]

\[ f'(x) = \frac{-4}{3(\sqrt[3]{x})^2}(x - \frac{1}{4}) \]

\( f(x) = 0 \iff x = 0, 1 \)

\( f(x) \) undefined \( \iff \mathbf{x} \)

\( f'(x) = 0 \iff x = 1/4 \)

\( f'(x) \) undefined \( \iff x = 0 \)

endpts: \( -1, 1 \)

crit. pts: \( x = 0, 1/4 \)

values: \( f(-1) = -2, f(0) = 0, f(1/4) = \frac{3\sqrt{2}}{8} \quad f(1) = 0 \)

(A) \( x = -1 \) (B) \( x = 0 \) (C) \( x = 1/4 \) (D) \( x = 0 \) (E) #

- Which \( x \) is an absolute maximum?
- Which \( x \) is an absolute minimum?
- Which \( x \) is a critical point but not an extreme?

(A) [-1, 1/4] (B) [-1, 0] (C) [1/4, 1] (D) [0, 1] (E) #

- Which is the one interval of increase?
- Which is the interval of decrease?