**Rolle’s Theorem** For any differentiable \( f \), any two distinct numbers \( a \) and \( b \), if \( f(a) = f(b) = 0 \) then \( f'(c) = 0 \) for some \( c \) between \( a \) and \( b \).

![Diagram](image)

**Proof.** If \( f \) is 0 on \([a, b]\) then \( f' = 0 \) for all \( c \) in \([a, b]\). Otherwise \( f(x) > 0 \) or \( f(x) < 0 \) somewhere in \([a, b]\). Consider the first case. In this case \( f \) has a maximum \( f(c) > 0 \) at some \( c \in (a, b) \). Thus \( c \) is a critical point. \( c \) is not an endpoint since \( c \in (a, b) \). \( f'(c) \) is not undefined since \( f \) is differentiable. Hence \( f'(c) = 0 \).

**Corollary** For any differentiable \( f \), if \( f \) has \( n + 1 \) zeros, then \( f' \) has \( n \) or more zeros. I.e., \( f \) has at most one more zero than \( f' \).

**Proof.** By the above theorem, between every two zeros of \( f \) there is a zero of \( f' \). Hence if \( f \) has three zeros, \( a, b, c \), then \( f' \) has two or more zeros, one between \( a, b \) and another between \( b, c \).

The derivative of a degree \( n \) polynomial has degree \( n - 1 \). Hence a degree \( n \) polynomial has at most one more zero than possible for a degree \( n - 1 \) polynomial. A degree 1 polynomial is a nonconstant straight line with exactly 1 root. A degree 2 polynomial has at most one more, i.e., at most 2 roots. ... A degree \( n \) polynomial has at most \( n \) roots.

More generally, for any secant from \((a, f(a))\) to \((b, f(b))\) there is a tangent at some \( c \) between \( a \) and \( b \) which is parallel to the secant, i.e.,

**Mean Value Theorem** For any differentiable \( f \) and any two distinct numbers \( a \) and \( b \),

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

for some \( c \) between \( a \) and \( b \).

\[
\frac{f(b) - f(a)}{b - a}
\]

is the slope of the secant \((a, f(a))\) to \((b, f(b))\) and it is also the average rate of change over \([a, b]\).

**Corollary** If \( f' \) is zero everywhere, then \( f \) is some constant \( C \): \( f(x) = C \) for all \( x \).

**Proof by contradiction.** Assume the hypothesis: \( f' \) is zero everywhere. Assume, for sake of a contradiction, that \( f \) is not constant. Thus \( f(b) \neq f(a) \) for some points \( a \) and \( b \).

Thus \( f(b) - f(a) \neq 0 \) and \( \frac{f(b) - f(a)}{b - a} \neq 0 \). By the Mean Value Theorem, there is a \( c \) between \( a \) and \( b \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Hence \( f'(c) \neq 0 \) which contradicts our hypothesis. Hence \( f \) is constant (the assumption that was not led to a contradiction).

**Corollary** For any differentiable functions \( f, g \), if \( f'(x) = g'(x) \), then \( f(x) = g(x) + C \) for some constant \( C \).

**Proof.** \( f' = g' \) implies \( f - g = f' - g' = 0 \). By the previous Corollary, this implies \( f - g = C \) for some constant \( C \). Hence \( f(x) = g(x) + C \).

**Definition** \( f \) is an antiderivative of \( g \) iff \( f' = g \) iff \( f \) is a solution to \( y' = g(x) \).

**Corollary** If \( f \) is a solution to \( y' = g(x) \), then every solution is \( f(x) + C \) for some constant \( C \). I.e., if \( f \) is an antiderivative of \( g \) then any other antiderivative of \( g \) is \( f(x) + C \) for some constant \( C \).

**Proof.** Suppose that \( f'(x) = g(x) \). If \( h \) is another solution, then \( h'(x) = g(x) \). Thus \( h'(x) = f'(x) \). By the previous Corollary, \( h \) and \( f \) differ by a constant: \( h(x) = f(x) + C \).

Find all functions \( y(x) \) such that \( y' = x^2 \).

\[
\begin{align*}
(?)' &= x^2 \\
(x^3)' &= 3x^2 \\
\left(\frac{1}{3}x^3\right)' &= x^2 \\
\end{align*}
\]

Hence every solution is \( \frac{1}{3}x^3 + C \) for some \( C \).

**Differential Equations**

- Solve for \( g \) given a differential equation and initial condition.
- Find \( f \) if \( f'(x) = 2\cos(x) \) and \( f(0) = 4 \).
  \[
  \begin{align*}
  (?)' &= 2\cos(x) \\
  \sin(x)' &= \cos(x) \\
  (2\sin(x))' &= 2\cos(x)
  \end{align*}
  \]
  Hence \( f(x) = 2\sin(x) + C \) for some \( C \) which we must find.
  \( f(0) = 4 \) implies \( 2\sin(0) + C = 4 \) implies \( 2(0) + C = 4 \) implies \( C = 4 \). Hence \( f(x) = 2\sin(x) + 4 \).
- Find \( s \) if \( a(x) = 2x \), \( s(0) = 0 \), \( v(1) = 2 \).
  Stated another way:
  - Find \( s \) if \( v'(x) = 2x \), \( v(1) = 2 \), \( s'(x) = v(x) \), \( s(0) = 0 \).
  - \( v(x) = ? \) Use the facts \( v'(x) = 2x \), \( v(1) = 2 \)
    \[
    \begin{align*}
    (A) \ x^2 & \quad (B) \ \frac{1}{2}x^2 & \quad (C) \ x^2 + 1 & \quad (D) \ x^2 + 2 & \quad (E) \ #
    \end{align*}
    \]
  - \( s(x) = ? \) Use the facts \( s'(x) = v(x) \), \( s(0) = 0 \)
    \[
    \begin{align*}
    (A) \ x^3 & \quad (B) \ \frac{1}{3}x^3 & \quad (C) \ x^3 + 1 & \quad (D) \ \frac{1}{3}x^3 + x & \quad (E) \ #
    \end{align*}
    \]
  - Prove that \( f(x) = \frac{1}{x^2} - x^3 \) has at most one zero in the interval \((0, \infty)\).
  Solution: \( f'(x) = (x^{-2})' - (x^3)' = -2x^{-3} - 3x^2 = \frac{-2}{x^3} - 3x^2 \).
  For \( x \in (0, \infty) \) this is negative and can’t be 0. By the Corollary, if \( f' \) has \( n \) zeros, then \( f \) has at most \( n + 1 = 0 + 1 = 1 \) zero in the interval \((0, \infty)\).