**Rolle's Theorem** For any differentiable \( f \), any two distinct numbers \( a \) and \( b \), if \( f(a) = f(b) = 0 \) then \( f'(c) = 0 \) for some \( c \) between \( a \) and \( b \).

**Proof.** If \( f \) is 0 on \([a, b]\) then \( f' = 0 \) for all \( c \) in \([a, b]\). Otherwise \( f(x) > 0 \) or \( f(x) < 0 \) somewhere in \([a, b]\). Consider the first case. In this case \( f \) has a maximum \( f(c) > 0 \) at some \( c \in (a, b) \). Thus \( c \) is a critical point. \( c \) is not an endpoint since \( c \in (a, b) \). \( f'(c) \) is not undefined since \( f \) is differentiable. Hence \( f'(c) = 0 \).
**Corollary** For any differentiable \( f \), if \( f \) has \( n + 1 \) zeros, then \( f' \) has \( n \) or more zeros. I.e., \( f \) has at most one more zero than \( f' \).

**Proof.** By the above theorem, between every two zeros of \( f \) there is a zero of \( f' \). Hence if \( f \) has three zeros, \( a, b, c \), then \( f' \) has two or more zeros, one between \( a, b \) and another between \( b, c \).

The derivative of a degree \( n \) polynomial has degree \( n - 1 \). Hence a degree \( n \) polynomial has at most one more zero than possible for a degree \( n - 1 \) polynomial. A degree 1 polynomial is a nonconstant straight line with exactly 1 root. A degree 2 polynomial has at most one more, i.e., at most 2 roots. ... A degree \( n \) polynomial has at most \( n \) roots.

More generally, for any secant from \((a,f(a))\) to \((b,f(b))\) there is a tangent at some \( c \) between \( a \) and \( b \) which is parallel to the secant, i.e.,
**Mean Value Theorem** For any differentiable $f$ and any two distinct numbers $a$ and $b$,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $c$ between $a$ and $b$.

The expression $\frac{f(b) - f(a)}{b - a}$ is the slope of the secant $(a, f(a))$ to $(b, f(b))$ and it is also the average rate of change over $[a, b]$. 
**Corollary** If \( f' \) is zero everywhere, then \( f \) is some constant \( C \): \( f(x) = C \) for all \( x \).

**Proof by contradiction.** Assume the hypothesis: \( f' \) is zero everywhere. Assume, for sake of a contradiction, that \( f \) is not constant. Thus \( f(b) \neq f(a) \) for some points \( a \) and \( b \). Thus \( f(b) - f(a) \neq 0 \) and \( \frac{f(b) - f(a)}{b - a} \neq 0 \). By the Mean Value Theorem, there is a \( c \) between \( a \) and \( b \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \). Hence \( f'(c) \neq 0 \) which contradicts our hypothesis. Hence \( f \) is constant (the assumption that is was not led to a contradiction).

**Corollary** For any differentiable functions \( f, g \), if \( f'(x) = g'(x) \), then \( f(x) = g(x) + C \) for some constant \( C \).

**Proof.** \( f' = g' \) implies \( (f - g)' = f' - g' = 0 \). By the previous Corollary, this implies \( f - g = C \) for some constant \( C \). Hence \( f(x) = g(x) + C \).
**Definition** $f$ is an antiderivative of $g$ iff $f' = g$ iff $f$ is a solution to $y' = g(x)$.

**Corollary** If $f$ is a solution to $y' = g(x)$, then every solution is $f(x) + C$ for some constant $C$. I.e., if $f$ is an antiderivative of $g$ then any other antiderivative of $g$ is $f(x) + C$ for some constant $C$.

**Proof.** Suppose that $f'(x) = g(x)$. If $h$ is another solution, then $h'(x) = g(x)$. Thus $h'(x) = f'(x)$. By the previous Corollary, $h$ and $f$ differ by a constant: $h(x) = f(x) + C$.

- Find all functions $y(x)$ such that $y' = x^2$.

  $(?)' = x^2$
  $(x^3)' = 3x^2$
  $(\frac{1}{3}x^3)' = x^2$

  Hence every solution is $\frac{1}{3}x^3 + C$ for some $C$. 

Differential Equations

Find \( f \) if \( f'(x) = 2 \cos(x) \) and \( f(0) = 4 \).

\[ (\sin(x))' = \cos(x) \]
\[ (2 \sin(x))' = 2 \cos(x) \]

Hence \( f(x) = 2 \sin(x) + C \) for some \( C \) which we must find.

\[ f(0) = 4 \implies 2 \sin(0) + C = 4 \implies 2(0) + C = 4 \]
implies \( C = 4 \). Hence \( f(x) = 2 \sin(x) + 4 \).

Find \( s \) if \( a(x) = 2x, \ s(0) = 0, \ v(1) = 2 \). Stated another way

Find \( s \) if \( v'(x) = 2x, \ v(1) = 2, \ s'(x) = v(x), \ s(0) = 0 \).

\[ v(x) = ? \quad \text{Use the facts} \quad v'(x) = 2x, \ v(1) = 2 \]
(A) \( x^2 \)  (B) \( \frac{1}{2} x^2 \)  (C) \( x^2 + 1 \)  (D) \( x^2 + 2 \)  (E) #

\[ s(x) = ? \quad \text{Use the facts} \quad s'(x) = v(x), \ s(0) = 0 \]
(A) \( x^3 \)  (B) \( \frac{1}{3} x^3 \)  (C) \( x^3 + 1 \)  (D) \( \frac{1}{3} x^3 + x \)  (E) #
Prove that \( f(x) = \frac{1}{x^2} - x^3 \) has at most one zero in the interval \((0, \infty)\).

Solution: \( f'(x) = (x^{-2})' - (x^3)' = -2x^{-3} - 3x^2 = \frac{-2}{x^3} - 3x^2 \). For \( x \in (0, \infty) \) this is negative and can’t be 0. By the Corollary, if \( f' \) has \( n = 0 \) zeros, then \( f \) has at most \( n + 1 = 0 + 1 = 1 \) zero in the interval \((0, \infty)\).

\( \square \)