Concavity and inflection

**Definition** For any function $f$ and any interval:
- $f$ is *concave up* on the interval (curving up $\cup$ like a cup) iff $f'$ is increasing,
- $f$ is *concave down* on the interval (curving down $\cap$ like a dome) iff $f'$ is decreasing,
- $(c, f(c))$ is an *inflection point* iff for points near $c$, $f$ is concave up $\cup$ on one side of $c$ and concave down $\cap$ on the other.

Concave up $\cup$ iff the graph lies above tangent lines.
Concave down \( \iff \) iff the graph lies below tangent lines.

At an inflection point, the graph is above the tangent on one side and below on the other.
- List the intervals of concavity and inflection points.

- There are inflection points at $x = -1, 0, 1$.

  - On $(1, \infty)$
    - (A) concave up
    - (B) concave down
    - (C) both
    - (D) neither

  - On $(-\infty, -1)$, the graph is (select one of the options above) --
  - On $(-1, 0)$, the graph is (select one of the options above) --
  - On $(0, 1)$, the graph is (select one of the options above) --

- Infl. pts.: $(-1, -1/2), (0, 0), (1, 1/2)$ (list both coordinates)
Note that

\( f'' > 0 \) iff \( f' \) is increasing iff the graph’s slopes are increasing iff the graph is curling up \( \uparrow \) iff the graph is concave up.

\( f'' < 0 \) iff \( f' \) is decreasing iff the graph’s slopes are decreasing iff the graph is curling down \( \downarrow \) iff the graph is concave down.

**Theorem** For a function \( f \) on an interval and a point \( c \):

- \( f''(x) > 0 \) implies \( f \) is concave up \( \uparrow \) on the interval
- \( f''(x) < 0 \) implies \( f \) is concave down \( \downarrow \) on the interval,
- \((c, f(c))\) an inflection point \( \Rightarrow f''(c) = 0 \) or \( f''(c) \) d.n.e.

To find all inflection points of \( f \),
- find all \( x \) such that \( f''(x) = 0 \) or d.n.e. and
- determine which of these are inflection points
**Theorem**  For differentiable $f$ and $a$ a critical point:

- If $f''(a) > 0$ then, at $x = a$, there is a
  (A) loc. min.  (B) loc. max.  (C) both  (D) neither

- If $f''(a) < 0$ then, at $x = a$, there is a
  (A) loc. min.  (B) loc. max.  (C) both  (D) neither
\[ y = f(x) = x^5 - 5x^4 \]
\[ y = x^4(x - 5), \quad \text{lead term } x^5, \quad \text{roots}, \quad x = 0, 5 \]
\[ y' = 5x^4 - 20x^3 = 5x^3(x - 4), \quad \text{lead term } 5x^4 \quad \text{crit. } x = 0, 4 \]
\[ y'' = 20x^3 - 60x^2 = 20x^2(x - 3), \quad \text{lead term } 20x^3 \quad \text{infl? } x = 0, 3 \]

**Values:**
\[ f(0) = 0, \quad f(5) = 0 \]
\[ f(3) = 3^4(3 - 5) = (-2)3^4 = -162, \quad f(4) = -4^4 = -256 \]

Conv. down on \((-\infty, 0), (0, 3)\) join these.
\[ \searrow \quad (-\infty, 3), \quad \nearrow \quad (3, \infty) \quad \text{(exclude endpoints)} \]

Infl. pts.: \(x = 3\) but not \(x = 0\)
More detailed analysis:

- \( y = f(x) = x^5 - 5x^4 \)
- \( y = x^4(x - 5), \) lead term \( x^5 \), roots, \( x = 0, 5 \)

- \( y' = 5x^4 - 20x^3 = 5x^3(x - 4), \) lead term \( 5x^4 \) crit. \( x = 0, 4 \)

Crit. pts (\( f' = 0 \) or d.n.e): \( x = 0 \) (hor.), 4 (hor.)

- \( y'' = 20x^3 - 60x^2 = 20x^2(x - 3), \) lead \( 20x^3 \) infl? \( x = 0, 3 \)

Infl. pts (\( f'' = 0 \) or d.n.e): \( x = 0 \) (not infl.), 3 (infl.)

Conv. down on \( (-\infty, 0), (0, 3) \) join these.

\[ f''(0) = 0 \] does not help. But the graph is concave down on both sides, hence \( f(0) = 0 \) is a loc. max.,

\( f''(4) > 0, \) \( f(4) = -4^4 \) loc. min.

Values: \( f(0) = 0, f(3) = -162, f(4) = -256, f(5) = 0 \)

Graph ...