Min/max word problems continued

The total surface area (sides+top+bottom) of a can (right circular cylinder) is 1000 square centimeters. Find the dimensions (radius and height) of the can with largest volume. Prove your answer. Curved surface area: \( S = 2\pi rh \)

Picture: Draw the picture and indicate the variables.

\[ \text{r} \quad \text{V = volume (max)} \]

\[ \text{h} \quad \text{surface area = 1000} \]

Given: List the given facts. Don’t use variables to name constants.

(A) \( V = \pi r^2 h, \ S = 1000 \)
(B) \( V = \pi r^2 h, \ S = 2\pi r^2 + 2\pi rh \)
(C) \( V = \pi r^2 h, \ 2\pi rh + 2\pi r^2 = 1000 \)
(D) \( \pi r^2 h = 1000, \ S = 2\pi r^2 + 2\pi rh \)
(E) #
One variable: Write the max/min variable in one other variable.

\[ \pi r^2 + \pi rh = 500 \]

\[ h = \frac{500 - \pi r^2}{\pi r} \]

\[ V = \pi r^2 h = \pi r^2 \frac{500 - \pi r^2}{\pi r} = r(500 - \pi r^2) = 500r - \pi r^3 \]

Critical points:

\[ \frac{dV}{dr} = 500 - 3\pi r^2. \quad \frac{dV}{dr} \text{ undef: never} \]

\[ \frac{dV}{dr} = 0 \iff 3\pi r^2 = 500 \iff r = \sqrt{\frac{500}{3\pi}} = 10 \sqrt{\frac{5}{3\pi}} \]

\[ h = \frac{500 - \pi r^2}{\pi r} = \frac{500 - \pi \frac{500}{3\pi}}{\pi \sqrt{\frac{500}{3\pi}}} \]

\[ = \frac{500(1 - \frac{1}{3})}{\pi \sqrt{\frac{500}{3\pi}}} = \frac{500(\frac{2}{3})}{\pi \sqrt{\frac{500}{3\pi}}} = \]

\[ = \frac{2}{\sqrt{\frac{500}{3\pi}}} \left(\frac{500}{3\pi}\right) = 2\sqrt{\frac{500}{3\pi}} = 20\sqrt{\frac{5}{3\pi}} \]
Answer: Answer in English with units but no introduced variables.
The radius is $10\sqrt{\frac{5}{3\pi}}$ cm, the height is $20\sqrt{\frac{5}{3\pi}}$ cm.

Prove your answer: \[
\frac{d^2 V}{dr^2} = (500 - 3\pi r^2)' = -6\pi r < 0
\]
A local max by the second derivative test. An absolute max since there is no other local extreme.
A rectangle of width $w$ and height $h$ is inscribed in an 8 inch diameter circle. Find the maximum area of the rectangle. In the homework problem you must find the dimensions. Prove your answer.

**Picture:** Draw the picture and indicate the variables.

![Diagram of a rectangle inscribed in a circle](image)

$A = \text{area (max)}$

**Given:** $w^2 + h^2 = 8^2$ \hspace{2em} $A = wh$

**One variable:** Write the max/min variable in one other variable.

(A) $A = wh$
(B) $A = w\sqrt{64 - w^2}$
(C) $A = h\sqrt{64 - h^2}$
(D) $A = w(64 - w^2)$
(E) $A = h(64 - h^2)$
Critical points: The domain is \( w \in [0, 8] \)

\[
\frac{dA}{dw} = \sqrt{64 - w^2} + w \frac{(-2w)}{2 \sqrt{64 - w^2}}
\]

\[
= \frac{(64 - w^2) - w^2}{\sqrt{64 - w^2}} = \frac{64 - 2w^2}{\sqrt{64 - w^2}} = \frac{2(32 - w^2)}{\sqrt{64 - w^2}}
\]

\[
\frac{dA}{dw} = 0 \iff w^2 = 32 = (2)16 \iff w = 4\sqrt{2}
\]

\[
\frac{dA}{dw} \text{ indef.: } 64 - x^2 = 0 \iff x = 8
\]

Endpts: \( x = 0, 8 \)

Critical pts: \( x = 0, 4\sqrt{2}, 8 \)

Values: \( A = w\sqrt{64 - w^2} \)

\[
A|_{w=0} = A|_{w=8} = 0
\]

\[
A|_{w=4\sqrt{2}} = w\sqrt{64 - w^2} \bigg|_{w=\sqrt{32}} = (\sqrt{32} \cdot \sqrt{32}) = 32
\]

Answer: Answer in English with units but no introduced variables. The maximum area is 32 square inches.
Prove your answer: It is an absolute max since the domain is finite and our critical point has the largest value.

If the question asked for the dimensions of the rectangle of maximum area instead of asking for the area, then answer would be: The width is \( 4\sqrt{2} \) inches and the height is \( 4\sqrt{2} \) inches.

Find the dimensions of the base of the box of greatest volume if the surface area including the top is 100 square inches and the length of the base is twice its width.

**Picture:** Draw the picture and indicate the variables

\[
\begin{align*}
V &= \text{volume (max)} \\
\text{surface area} &= 100 \\
2w & \quad w & \quad h
\end{align*}
\]

In the previous groupwork problem the volume was fixed and we wanted the minimum surface area. Here the surface area is fixed and we want the greatest volume.
**Domain:** Allowing degenerate boxes, the minimal width is \( w = 0 \). The width is maximal when the height \( h = 0 \). In this case the sides have no area and the surface area is just the top plus the bottom = \( w(2w) + w(2w) = 4w^2 = 100 \). Hence \( w^2 = 25 \) and \( w = 5 \). Thus the domain for \( w \) is \([0, 5]\). At either of these endpoints, when either \( w \) or \( h = 0 \), the volume will be 0, i.e., the minimum volume rather than the wanted maximum volume. Endpoints usually give the opposite of what is wanted. For word problems, the critical point of interest is usually where the derivative is 0 rather than where it does not exist or at the endpoints.