Min/max word problems continued

- Find the width of the isosceles triangle of largest area with a perimeter of 12 meters. Prove your answer.

**Picture:** Draw the picture and indicate the variables.

![isosceles triangle](image)

- perimeter = 12 meters
- A = area (max)
- Want the width 2x, w or x?

**Given:** List the given facts.
(A) \( A = \frac{1}{2}xh, \ P = 2x + 2h \)
(B) \( A = \frac{1}{2}xh, \ P = 2x + 2\sqrt{x^2 + h^2} \)
(C) \( A = xh, \ P = 12 \)
(D) \( A = xh, \ 2x + 2\sqrt{x^2 + h^2} = 12 \)
(E) #

**One variable and domain:** Domain: \( h \in [0, 6] \)

\[
x + \sqrt{x^2 + h^2} = 6 \\
\sqrt{x^2 + h^2} = 6 - x \\
x^2 + h^2 = 36 - 12x + x^2 \\
h^2 = 36 - 12x \\
12x = 36 - h^2 \\
x = \frac{1}{12}(36 - h^2) \\
A = xh = \frac{1}{12}(36 - h^2)h = \frac{1}{12}(36h - h^3) \\
\]

**Diff.** \( \frac{dA}{dh} = \frac{1}{12}(36 - 3h^2) = \frac{3}{12}(12 - h^2) = \frac{1}{4}(12 - h^2) \)

**Critical points:** Differentiate; set the derivative to zero; solve.

Endpoints of \( [0, 6] \): \( h = 0, 6 \)

\( \frac{dA}{dh} \) d.n.e. \( \star \).

\[
\frac{dA}{dh} = 0 \text{ iff } 12 - h^2 = 0 \text{ iff } h = \sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}
\]

Critical points: \( h = 0, 2\sqrt{3}, 6 \)

Values: \( A|_{h=0} = 0, \ A|_{h=6} = 0, \ A|_{h=2\sqrt{3}} > 0 \)

Max when \( h = 2\sqrt{3} \).

\[
x = \frac{1}{12}(36 - h^2) = \frac{1}{12}(36 - 12) = 3 - 1 = 2
\]

Answer: Height = \( 2\sqrt{3} \) meters; width = \( 2x = 4 \) meters.

**Proof:** This gives an absolute max since it has the max value of the three critical points.

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- You have a 6 inch diameter circle of paper. You remove a pie-shaped wedge and form the remaining paper into a cone-shaped cup of some height \( h \) and radius \( r \). Find height and radius which gives the cup of largest volume. The volume of a cone is \( \frac{1}{3} \) the volume of the cylinder of the same radius and height. \( V = \frac{1}{3} \pi r^2 h \).

**Prove your answer.**

**Picture:** Draw the picture and indicate the variables.

![cone](image)

- Paper radius = 6 inches
- V = volume (max)
- Want \( r, h \)

**Given:** List the given facts.
\( h^2 + r^2 = 6^2 \)
\( V = \frac{1}{3} \pi r^2 h \)

**One variable:**
(A) \( V = \frac{1}{3} \pi r^2 \sqrt{36 - r^2} \)
(B) \( V = \frac{1}{3} \pi (36 - h^2)h \)
(C) \( V = \frac{1}{3} \pi r^2 h \)
(D) \( V = \frac{1}{3} \pi r(36 - h^2)\sqrt{36 - r^2} \)
(E) #

**Domain:** \( [0, 6] \) \( V = \frac{1}{3} \pi (36 - h^2)h = \frac{3\pi}{3}(36h - h^3) \)

**Diff.** \( \frac{dV}{dh} = \frac{3\pi}{3}(12 - h^2) = \pi(12 - h^2) \)

**Critical points:** Differentiate; set the derivative to zero; solve.

Endpoints: domain = \( [0, 6] \): \( h = 0, 6 \)

\( \frac{dV}{dh} = 0 \text{ iff } h^2 = 12 \text{ iff } h = \sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3} \)

\( \frac{dV}{dh} \) undefined: \( \star \).

Critical points: \( r = 0, 2\sqrt{6}, 6 \)

\[
r^2 = 6^2 - h^2 = 36 - (4)(3) = 24
\]

\[
r = 2\sqrt{6}
\]

Answer: For max volume, the height is \( 2\sqrt{3} \) inches. The radius is \( 2\sqrt{6} \) inches.

**Proof:** \( \frac{d^2V}{dh^2} = \frac{d}{dh} \pi(12 - h^2) = -2\pi h < 0 \) By the second derivative test, we have a local max. and, being the only local extreme, it is an abs. max. \( \square \)