**Math 241  Lecture 36**

**RECALL** \( \int_a^b f(x) \, dx \) is the signed area between the \( x \)-axis and \( f(x) \) and between the points \( a \) and \( b \).

\( \int_a^b f(x) \, dx \) is the integral, \( x \) is the variable of integration, \( f(x) \) is the integrand.

**Upper/lower sums**

In order to approximate the definite integral we can divide the line segment \([a, b]\) into equal subsegments and from each segment extend a rectangle to the function’s maximum (minimum) value on that segment. The sum of the signed areas of these rectangles is an upper sum (lower sum).

Note, a lower sum is \( \leq \int_a^b f(x) \, dx \leq \) an upper sum.

Dividing into smaller subsegments gives better estimates.

- Estimate \( \int_1^7 \sqrt{x} \, dx \) with an upper sum of three rectangles.
  - The rectangle endpoints are \( 1 = \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3} = 2 \). The rectangle widths are \( \frac{1}{3} \).
  - Leave answer as an unsimplified sum of products.

Upper sum: \( \sqrt{\frac{3}{3}} \left( \frac{1}{3} \right) + \sqrt{\frac{4}{3}} \left( \frac{1}{3} \right) + \sqrt{2} \left( \frac{1}{3} \right) \)

Lower sum: \( \sqrt{\frac{1}{3}} \left( \frac{1}{3} \right) + \sqrt{\frac{4}{3}} \left( \frac{1}{3} \right) + \sqrt{\frac{2}{3}} \left( \frac{1}{3} \right) \)

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**Definite Integral Rules**

Find the following integrals by drawing the picture.

- \( \int_{-3}^{0} \sqrt{9-x^2} \, dx = \)
- \( \int_{-1}^{1} x \, dx = \)
  - (A) -1 (B) -\frac{1}{2} (C) 0 (D) \frac{1}{2} (E) 1

**Definite Integration Limit Rules**

<table>
<thead>
<tr>
<th>Integral</th>
<th>Expression</th>
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<tbody>
<tr>
<td>( \int_a^a f(x) , dx = 0 )</td>
<td></td>
</tr>
<tr>
<td>( \int_a^b f(x) , dx = -\int_b^a f(x) , dx )</td>
<td></td>
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<tr>
<td>( \int_a^b f(x) , dx + \int_b^c f(x) , dx = \int_a^c f(x) , dx ) for any order of ( a, b, c ).</td>
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In the above picture,

\( \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx \)

is clearly true but so is

\( \int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx \).

- \( \int_a^a f(x) \, dx = 2 \) and \( \int_a^b f(x) \, dx = 1 \), find \( \int_a^b f(x) \, dx \)
  - (A) -1 (B) -\frac{1}{2} (C) 0 (D) \frac{1}{2} (E) 1

**Integration Linearity Rules**

\( \int_a^b f(x) \, dx = \int_a^b f(t) \, dt = \int_a^b f(y) \, dy = \ldots \)

\( \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx \) for a constant \( c \)

\( \int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \)

\( f(x) \leq g(x) \implies \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \) for \( a \leq b \).

- \( \int_1^0 g(x) \, dx = 1/2, \ \int_1^0 [-2g(x)] \, dx = ? \)
  - (A) -1 (B) -\frac{1}{2} (C) 0 (D) \frac{1}{2} (E) 1
- \( \int_1^0 f(x) \, dx = -1, \int_1^0 g(x) \, dx = 1/2, \ \int_1^0 [f(x) + g(x)] \, dx = ? \)
  - (A) -1 (B) -\frac{1}{2} (C) 0 (D) \frac{1}{2} (E) 1