**Definition** $A$ is the average of $f$ on $[a, b]$ if

$$\int_{a}^{b} A \, dx = \int_{a}^{b} f(x) \, dx,$$

i.e., the area under $A$ is the same as the area under $f$.

Think of the area under $f$ as a 2-dimensional version of butter in a cup bounded by $a, b$ and the $x$-axis. When melted the hills will flow into the valleys and resulting top will be a level line segment whose height is the average.
To calculate $A$ note that $\int_{a}^{b} A \, dx = A(b - a)$ Hence

$\int_{a}^{b} A \, dx = \int_{a}^{b} f(x) \, dx$ implies

$A(b - a) = \int_{a}^{b} f(x) \, dx$ implies

$A = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx$

Find the following averages.

1. Average of $x$ on $[0, 1]$
   (A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

2. Average of $x$ on $[-1, 0]$
   (A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

3. Average of $x$ on $[-1, 1]$
   (A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1
Average of \( \sqrt{9 - x^2} \) on \([-3, 3]\)

\[
A = \frac{1}{3 - (-3)} \int_{-3}^{3} \sqrt{9 - x^2} \, dx
= \frac{1}{6} \left( \frac{1}{2} \pi (3)^2 \right) = \frac{1}{12} \pi 9 = \frac{3\pi}{4}
\]
**Fundamental Theorems**

**Evaluation Notation**  For any $f$, $[f(x)]_a^b = f(b) - f(a)$.

$[c]_a^b = c - c = 0$

$[cf(x)]_a^b = cf(b) - cf(a) = c(f(b) - f(a)) = c[f(x)]_a^b$.

$[f(x) + g(x)]_a^b = (f(b) + g(b)) - (f(a) + g(a))$

$= f(b) + g(b) - f(a) - g(a)$

$= f(b) - f(a) + g(b) - g(a) = [f(x)]_a^b + [g(x)]_a^b$

Hence the usual linearity rules apply.

$[cf(x)]_a^b = c[f(x)]_a^b$

$[f(x) \pm g(x)]_a^b = [f(x)]_a^b \pm [g(x)]_a^b$

$\textbf{Example: } [2x^3 + 1]_1^2 = 2[x^3]_1^2 + [1]_1^2 = 2(2^3 - 1^3) + 0 = 2(7) = 14$
**Fundamental Theorem II** If $F$ is an antiderivative of a continuous function $f$, then $\int_{a}^{b} f(x) \, dx = [F(x)]_{a}^{b}$.

- $\int_{1}^{3} 2x \, dx = [x^2]_{1}^{3} = ?$
  (A) 4  (B) 6  (C) 8  (D) 9  (E) 10

- $\int_{1}^{3} x \, dx = \left[ \frac{x^2}{2} \right]_{1}^{3} = \frac{1}{2} [x^2]_{1}^{3} = ?$
  (A) 4  (B) 6  (C) 8  (D) 9  (E) #
**Fundamental Theorem I** For continuous $f$,

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x) \quad \therefore \quad \int_a^x f(t) \, dt \text{ is an antiderivative of } f.
\]

The bottom limit is a constant, the top is a single variable. The dummy variable $t$ is replaced by the top variable $x$.

\[
(\int_a^1 \frac{1}{\sqrt{t + 1}} \, dt)' \neq \frac{1}{\sqrt{x + 1}}.
\]

The variable needs to be on top.

\[
(\int_a^1 \frac{1}{\sqrt{t + 1}} \, dt)' = - (\int_1^x \frac{1}{\sqrt{t + 1}} \, dt)' = \frac{-1}{\sqrt{x + 1}}
\]

When the variable $x$ is replaced by an inner function $x^2$, the chain rule applies and one must multiply by the derivative $2x$ of the inner function.
\[ (\int_1^{x^2} \frac{1}{\sqrt{t+1}} \, dt)' = \frac{1}{\sqrt{x^2 + 1}} (2x) = \frac{2x}{\sqrt{x^2 + 1}} \]

\[ \frac{d}{dx} \int_1^{\cos x} \sqrt{t} \, dt = \sqrt{\cos x} (\cos x)' \]
\[ = \sqrt{\cos x} (-\sin x) = -\sin x \sqrt{\cos x} \]

\[ \frac{d}{dx} \int_{\cos x}^{c^3} \sqrt{t} \, dt = -\frac{d}{dx} \int_{c^3}^{\cos x} \sqrt{t} \, dt = \]
\[ = -(\sqrt{\cos x} (-\sin x)) = \sin x \sqrt{\cos x} \]