Geometric Area between a graph and the x-axis

For areas above the x-axis, the geometric area (or total area or unsigned area) is the signed area. For areas below the x-axis, the geometric area is the negative of the signed area. The geometric area between a function and the x-axis is the negative of the signed areas where it is negative plus the signed areas where it is positive. To get the geometric area, split the domain into its key intervals then total the absolute values of the definite integral on these intervals.

Find the geometric area between y = x and the x-axis over [-1, 1]

\[-\int_{-1}^{0} x \, dx + \int_{0}^{1} x \, dx = -\left[ \frac{x^2}{2} \right]_{-1}^{0} + \left[ \frac{x^2}{2} \right]_{0}^{1} = -\frac{-1}{2} + \frac{1}{2} = 1.\]
- Total area between $y = \cos x$ and $x$-axis over $[0, \frac{3\pi}{2}]$. $\cos x$ is positive over $[0, \frac{\pi}{2}]$; negative over $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Hence the total (geometric) area is

$$
\int_{0}^{\pi/2} \cos x \, dx - \int_{\pi/2}^{3\pi/2} \cos x \, dx
$$

$$
= [\sin x]\bigg|_{0}^{\pi/2} - [\sin x]\bigg|_{\pi/2}^{3\pi/2}
$$

$$
= (\sin(\pi/2) - \sin(0)) - (\sin(3\pi/2) - \sin(\pi/2))
$$

$$
= (1 - 0) - (-1 - 1) = 1 + 1 + 1 = 3
$$
Recall: $\int \sin x \, dx = -\cos x + C$, \quad $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

- Total area between $y = \sin x$ and $x$-axis over $[\frac{\pi}{4}, \frac{3\pi}{2}]$.

Integral
(A) $\int_{\pi/4}^{\pi/2} \sin x \, dx + \int_{\pi/2}^{3\pi/2} \sin x \, dx$
(B) $\int_{\pi/4}^{\pi/2} \sin x \, dx - \int_{\pi/2}^{3\pi/2} \sin x \, dx$
(C) $|\int_{\pi/4}^{\pi} \sin x \, dx| + |\int_{\pi}^{3\pi/2} \sin x \, dx|$
(D) $\int_{\pi/4}^{\pi} \sin x \, dx - \int_{\pi}^{3\pi/2} \sin x \, dx$ \quad (E) #

Value
(A) $2 - \frac{3}{\sqrt{2}}$  \quad (B) $2 - \frac{1}{\sqrt{2}}$  \quad (C) $2 + \frac{1}{\sqrt{2}}$  \quad (D) $2 + \frac{3}{\sqrt{2}}$  \quad (E) #
Find the total area between \( y = x^3 + x^2 - 2x \) and the \( x \)-axis over \([-1, 1]\) and over \([0, 2]\).

\[
x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x + 2)(x - 1)
\]

Lead. term: \( x^3 \)

Roots: \( x = -2, 0, 1 \), all of degree 1.

Intervals: – on \((−∞, −2]\), + on \([-2, 0]\), – on \([0, 1]\), + on \([1, ∞)\)

Graph:

\[
\int x^3 + x^2 - 2x \, dx = \frac{x^4}{4} + \frac{x^3}{3} - x^2 + C
\]
Area between \( y = x^3 + x^2 - 2x \) and x-axis over \([-1, 1]\).

\[
= \int_{-1}^{0} y \, dx - \int_{0}^{1} y \, dx
\]

\[
= \left[ \frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_{-1}^{0} - \left[ \frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_{0}^{1}
\]

\[
= [(0) - \left( \frac{(-1)^4}{4} + \frac{(-1)^3}{3} - (-1)^2 \right)] - \left[ \left( \frac{1^4}{4} + \frac{1^3}{3} - 1^2 \right) - (0) \right]
\]

\[
= [-(\frac{1}{4} + \frac{-1}{3} - 1)] - [\frac{1}{4} + \frac{1}{3} - 1]
\]

\[
= -\frac{1}{4} + \frac{1}{3} + 1 - \frac{1}{4} - \frac{1}{3} + 1 = \frac{3}{2}
\]
Area between $y = x^3 + x^2 - 2x$ and the x-axis over $[0, 2]$

(A) $- \int_0^1 x^3 + x^2 - 2x \, dx + \int_1^2 x^3 + x^2 - 2x \, dx$

(B) $- \int_0^1 x^3 + x^2 - 2x \, dx - \int_1^2 x^3 + x^2 - 2x \, dx$

(C) $\int_0^1 x^3 + x^2 - 2x \, dx + \int_1^2 x^3 + x^2 - 2x \, dx$

(D) $\int_0^1 x^3 + x^2 - 2x \, dx - \int_1^2 x^3 + x^2 - 2x \, dx$  \hspace{1cm} (E) #

(A) $2/3$ \hspace{1cm} (B) $3/2$ \hspace{1cm} (C) $7/3$ \hspace{1cm} (D) $7/2$ \hspace{1cm} (E) #
Integral solutions of differential equations

For the differential equation
\[ y' = f(x), \quad y(a) = b \]
the integral solution is
\[ y = \int_a^x f(t) \, dt + b \]

It is a solution since
\[
\begin{align*}
    y' &= (\int_a^x f(t) \, dt)' + 0 = f(x) \quad \text{and} \\
    y(a) &= \int_a^a f(t) \, dt + b = 0 + b = b
\end{align*}
\]

To get a complete solution you need to find the integral.

Find the integral solution and the complete solution for
\[ y' = \frac{1}{x}, \quad y(1) = 50 \]

Integral solution: \[ y = \int_1^x \frac{1}{t} \, dt + 50 \]

Complete solution: \[ \ln(x) + 50 \]

By the dummy variable rule, we can also write
\[ \int_1^x \frac{1}{u} \, du + 50y = \int_1^x \frac{1}{s} \, ds + 50 \quad \text{but not} \quad y = \int_1^x \frac{1}{x} \, dx + 50 \]
For the differential equation 
\[ y' = f(x), \quad y(a) = b \]
the integral solution is
\[ y = \int_a^x f(t) \, dt + b \]

Find the integral solution: \( y' = \tan x, \quad y(0) = 1 \)

(A) \( \int_0^t \tan t \, dt \)  
(B) \( \int_1^t \tan x \, dx \)  
(C) \( \int_0^x \tan u \, du + 1 \)

(D) \( \int_1^x \tan t \, dt \)  
(E) \#