Areas (geometric area) between curves
If \( f \) and \( g \) are curves with and \( f(x) \geq g(x) \) on \([a, b]\) then the area (geometric area or total area) between the graphs is

\[
\int_a^b (f(x) - g(x)) \, dx
\]

This formula is always correct, even for areas below the \( x \)-axis.

If \( f(x) \leq g(x) \), the area is

\[
\int_a^b (g(x) - f(x)) \, dx = -\int_a^b (f(x) - g(x)) \, dx = \int_a^b (f(x) - g(x)) \, dx
\]

As long as \( f \) and \( g \) are continuous and don’t cross on the interval \([a, b]\), then one of \( f(x) \geq g(x) \) or \( f(x) \leq g(x) \) will be true on the interval. In either case the area will be

\[
\left| \int_a^b (f(x) - g(x)) \, dx \right|
\]

As long as the functions don’t intersect over an interval, just calculate the integral. If the answer is negative, drop the negation sign.

**Enclosed Area Calculation Procedure**

*Intersects:* To find where the graphs intersect, set the functions equal to each other and solve for \( x \).

*Areas:* In each interval between intersects, calculate the area \( \int_a^b (f(x) - g(x)) \, dx \) over the interval. Then total the absolute values of these integrals.

- Find the enclosed area between
  \( y = 1 - x^2 \) and \( y = x - 1 \)

  *Intersects:*
  \( x - 1 = 1 - x^2 \) iff \( x^2 + x - 2 = 0 \)
  \( (x + 2)(x - 1) = 0 \) iff \( x = -2, 1 \)

  *Integral(s):*
  \[
  \int_{-2}^{1} [(x - 1) - (1 - x^2)] \, dx = \int_{-2}^{1} (x^2 + x - 2) \, dx
  \]
  \[
  = \left[ \frac{x^3}{3} + \frac{x^2}{2} - 2x \right]_{-2}^{1}
  \]
  \[
  = \left[ \left( \frac{1}{3} \right)^3 + \frac{1}{2} - 2 \right] - \left[ \left( \frac{-2}{3} \right)^3 + \frac{(-2)^2}{2} - 2(-2) \right]
  \]
  \[
  = \left( \frac{1}{3} + \frac{1}{2} - 6 \right) = 3 + \frac{1}{2} - 8 = \frac{1}{2} - \frac{10}{2} = -\frac{9}{2}
  \]

  *Area(s):*
  area = \( \left| \int_{-2}^{1} [(x - 1) - (1 - x^2)] \, dx \right| = \left| -\frac{9}{2} \right| = \frac{9}{2} \)

- Find the enclosed area (finite area) between the following three curves.
  \( y = \sqrt{x+1} \), \( y = -\sqrt{x+1} \), \( y = 5 - x \)

  *Intersects:*
  \( \pm \sqrt{x+1} = 5 - x \) iff \( x + 1 = (5 - x)^2 \)
  \( \text{iff} \ x + 1 = 25 - 10x + x^2 \iff x^2 - 11x + 24 = 0 \)
  \( \text{iff} \ (x - 3)(x - 8) = 0 \) iff \( x = 3, 8 \)

  \( \sqrt{x+1} = -\sqrt{x+1} \) iff \( x = ? \)
  (A) -1 (B) 0 (C) 1 (D) 3 (E) #

Divide the enclosed area into two parts with \( A \) the part which is left of the vertical line through (3,2) and \( B \) the part which is right of that vertical line.

The area of part \( B \) is \( \int_3^8 (5 - x) - (-\sqrt{x+1}) \, dx \).

- Find the integral for \( A \).
  (A) \( \int_3^8 \sqrt{x+1} - (-\sqrt{x+1}) \, dx \)
  (B) \( \int_2^3 \sqrt{x+1} - (-\sqrt{x+1}) \, dx \)
  (C) \( \int_3^5 \sqrt{x+1} - (-\sqrt{x+1}) \, dx \)
  (D) \( \int_2^3 \sqrt{x+1} - (-\sqrt{x+1}) \, dx \)
  (E) #

Alternately, we can regard these curves as functions of \( y \).

- Find the integral ranging over \( y \).
  (A) \( \int_{y=\sqrt{x+1}}^{y=5-x} \sqrt{x+1} \, dy \)
  (B) \( \int_{y=-\sqrt{x+1}}^{y=5-x} \sqrt{x+1} \, dy \)
  (C) \( \int_{y=\sqrt{x+1}}^{y=5-x} (y^2 - 1) \, dy \)
  (D) \( \int_{y=-\sqrt{x+1}}^{y=5-x} (y^2 - 1) \, dy \)
  (E) #
GRAPHS WITH $x$-DEPENDENT, $y$-INDEPENDENT

When thinking of $x$ as a function $x(y)$ of $y$, $y$ is the independent variable ranging freely over the $y$-axis. $x(y)$ is the $x$-coordinate of the graph point $(x(y), y)$. When $x$ is a function of $y$, the areas can be calculated as integrals of $x$ with respect to $y \int_a^b x(y) \, dy$.

It is often easier to visualize by swapping the axes: putting $y$ on the horizontal axis and $x$ on the vertical.

Write the area as an integral ranging over $y$.

(A) $\int_{-8}^3 (5 - y) - \sqrt{y + 1} \, dy$
(B) $\int_{-1}^8 (5 - y) - \sqrt{y + 1} \, dy$
(C) $\int_{-3}^2 (5 - y) - (y^2 - 1) \, dy$
(D) $\int_{2}^3 (5 - y) - (y^2 - 1) \, dy$
(E) #