321 For each 3-dimensional solid, find the volume.

2(7) A solid lies between the planes \( x = -1 \) and \( x = 1 \) (in 2-dimensional space, these are vertical lines; in 3-dimensional space, these are planes perpendicular to the \( x \)-axis). The cross-sections perpendicular to the \( x \)-axis are circles whose diameters run from \( y = x^2 \) to \( y = 2 - x^2 \).

The volume is \( \pi \) times a fraction with a 2-digit numerator and 2-digit denominator.

For the circle at the cross-section at \( x \), let \( r(x), d(x), A(x) \) be the radius, diameter, and area.

The diameter is
\[
d(x) = (2 - x^2) - x^2 = 2 - 2x^2 = 2(1 - x^2).
\]

The area of a circle of diameter \( d \) is
\[
A = \pi r^2 = \pi \left( \frac{d}{2} \right)^2 = \frac{1}{4} \pi d^2.
\]

Thus the area of the circle at cross-section \( x \) is
\[
A(x) = \frac{1}{4} \pi d(x)^2 = \frac{1}{4} \pi (2(1 - x^2))^2 = \pi (1 - 2x^2 + x^4).
\]

The volume is
\[
\int_{-1}^{1} A(x) \, dx = \int_{-1}^{1} \pi (1 - 2x^2 + x^4) \, dx
\]
\[
= \pi \left[ x - \frac{2}{3} x^\frac{3}{2} + \frac{1}{5} x^\frac{5}{2} \right]_{-1}^{1}
\]
\[
= \pi \left[ (1 - 2 \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}}) - ((-1) - 2(-\frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{5}}) \right]
\]
\[
= \frac{\pi}{2} \left[ (1 - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{5}}) - (-1 + \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{5}}) \right]
\]
\[
= \frac{\pi}{2} \left[ (1 - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{5}}) + (1 - \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{5}}) \right]
\]
\[
= \pi \left[ 2 - \frac{4}{\sqrt{2}} + \frac{2}{\sqrt{5}} \right] = \pi \left[ 2 - \frac{4}{\sqrt{2}} + \frac{2}{\sqrt{5}} \right] = \frac{16}{15} \pi.
\]

4(7) A solid lies between the planes \( x = -1 \) and \( x = 1 \). The cross-sections perpendicular to the \( x \)-axis are squares whose diagonals run from \( y = -\sqrt{1 - x^2} \) to \( y = \sqrt{1 - x^2} \).

The volume is a 3 symbol fraction, checksum=11.

For the square at the cross-section at \( x \), let \( d(x), A(x) \) be the diagonal length and area.

The diagonal length is
\[
d(x) = 2\sqrt{1 - x^2}.
\]

If a square has side \( s \) and diagonal \( d \) then,
\[
s^2 + s^2 = d^2, \quad 2s^2 = d^2, \quad s^2 = \frac{1}{2}d^2.
\]

The area of a square with diagonal length \( d \) is
\[
A = s^2 = \frac{1}{2}d^2.
\]

Thus the area of the square at cross-section \( x \) is
\[
A(x) = \frac{1}{2} d(x)^2 = \frac{1}{2} (2\sqrt{1 - x^2})^2 = 2 (1 - x^2)
\]

The volume is
\[
\int_{-1}^{1} A(x) \, dx = \int_{-1}^{1} 2 (1 - x^2) \, dx
\]
\[
= 2 \left[ x - \frac{1}{3} x^3 \right]_{-1}^{1} = 2 \left[ (1 - \frac{1}{3}) - ((-1) - \frac{1}{3}) \right]
\]
\[
= 2 \left[ 1 - \frac{1}{3} + 1 - \frac{1}{3} \right] = 2 \left[ 2 - \frac{2}{3} \right] = 2 \frac{4}{3} = \frac{8}{3}.
\]

8(7) The solid’s base is the unit circle \( x^2 + y^2 \leq 1 \). For each \( y \) between -1 and 1, the cross-section perpendicular to the \( y \)-axis is an isosceles right triangle with one leg on the unit circle at the base. Find the volume.

For the triangle at cross-section \( y \), let \( b(y), h(y), A(y) \) be the base and height length and area.

For a point \((x, y)\) on the unit circle, \( x^2 + y^2 = 1 \). Given \( y \), solving for \( x \) gives \( x = \pm \sqrt{1 - y^2} \).

Hence the base of the triangle at cross-section \( y \) runs from \(-\sqrt{1 - y^2}\) to \(\sqrt{1 - y^2}\).

Thus base length is \( b(y) = 2 \sqrt{1 - y^2} \).

Since the triangle is isosceles this is also the height \( h(x) \).

The area of the triangle at cross-section \( y \) is
\[
A(y) = \frac{1}{2} b(y) h(y) = \frac{1}{2} \left[ 2 \sqrt{1 - y^2} \right] \left[ 2 \sqrt{1 - y^2} \right]
\]
\[
= 2 (1 - y^2)
\]

The volume is
\[
\int_{-1}^{1} A(y) \, dy = \int_{-1}^{1} 2 (1 - y^2) \, dy
\]
\[
= 2 \left[ y - \frac{1}{3} y^3 \right]_{-1}^{1} = 2 \left[ (1 - \frac{1}{3}) - ((-1) + \frac{1}{3}) \right]
\]
\[
= 2 \left[ 1 - \frac{1}{3} + 1 - \frac{1}{3} \right] = 2 \left[ 2 - \frac{2}{3} \right] = 2 \frac{4}{3} = \frac{8}{3}.
\]