Volumes using disks and shells continued

Use the disc method to find the volume of a doughnut of outer radius \( R \) and inner radius \( r \).

The red circle with equation \( x^2 + y^2 = r^2 \) is rotated around the line \( y = R \) forming a doughnut. Solving for \( y \) gives \( y = \pm \sqrt{r^2 - x^2} \). For each \( x \in [-r, r] \), the vertical black cross section at \( x \) of length \( 2y \) traces out a washer when rotated around the line \( y = R \).

**Inner radius**: \( r_1 = \) the distance between \( y \) and \( R \),

**Inner disk area**: \( \pi r_1^2 = \pi (R - \sqrt{r^2 - x^2})^2 \)

**Outer radius**: \( r_2 = R + y = R + \sqrt{r^2 - x^2} \)

\[ r_1 = R - y \]

\[ r_2 = R + y \]

**Outer disk area**: \( \pi r_2^2 = \pi (R + \sqrt{r^2 - x^2})^2 \)

**Volume**: \( \int_{-r}^{r} \pi r_2^2 - \pi r_1^2 \, dx = \pi \int_{-r}^{r} r_2^2 - r_1^2 \, dx \)

\[ = \pi \int_{-r}^{r} (R + \sqrt{r^2 - x^2})^2 - (R - \sqrt{r^2 - x^2})^2 \, dx \]

\[ = \pi \left[ R^2 + 2R \sqrt{r^2 - x^2} + (r^2 - x^2) \right] - \left[ R^2 - 2R \sqrt{r^2 - x^2} + (r^2 - x^2) \right] \, dx \]

\[ = 4 \pi \int_{-r}^{r} 2R \sqrt{r^2 - x^2} \, dx \]

\[ = 4 \pi R \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx \]

\[ = 4 \pi R \left( \frac{\pi r^2}{2} \right) \]

\( = 2 \pi R \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx \)

\[ = 2 \pi R^2 r^2 = (2 \pi R)(\pi r^2) \]

Use the shell method to find the volume of a doughnut of outer radius \( R \) and inner radius \( r \).

The red circle with equation \( x^2 + y^2 = r^2 \) is rotated around the line \( y = R \) forming a doughnut.

For each \( y \in [-r, r] \), the horizontal brown cross section of height \( y \) and length \( 2x \) traces out a cylindrical shell when rotated around the line \( y = R \).

**Radius**: \( = \) the distance between \( y \) and \( R \), \( = R - y \).

**Height (length)**: \( 2x = 2\sqrt{r^2 - y^2} \)

**Shell area**: \( 2\pi(\text{shell radius})(h) = 2\pi(R-y)(2\sqrt{r^2 - y^2}) \)

**Volume**: \( \int_{r}^{R} 2\pi rh \, dy = \int_{-r}^{r} 2\pi(R-y)(2\sqrt{r^2 - y^2}) \, dy \)

\[ = 4\pi \int_{-r}^{r} R\sqrt{r^2 - y^2} \, dy - 4\pi \int_{-r}^{r} \sqrt{r^2 - y^2} y \, dy \]

\[ u = r^2 - y^2, \quad du = -2ydy, \quad -du/2 = ydy \]

\[ = 4\pi R \int_{-r}^{r} \sqrt{1 - u^2} \, du - 4\pi \int_{r(-r)}^{r(-r)} \sqrt{u} \, du/2 \]

\[ = 4\pi R(\frac{\pi r^2}{2}) - 4\pi \int_{0}^{\sqrt{r^2}} \sqrt{u} \, du/2 = 2\pi R\pi r^2 - 0 = 2\pi^2 R^2 \]

\( 2\pi^2 R^2 = (2\pi R)(\pi r^2) \)

The volume of a can of height \( 2R \) and radius \( r \) is the can you get by cutting the doughnut loop and straightening out the doughnut into a long cylinder.