1 more related rate, min-max problem.
Problems on last common final.
4 limit problems
5 derivatives
1 tangent line
1 implicit derivative tangent line
1 max/min word problem
1 related rate problem
1 graphing problem with critical points, inflection
points, intervals of increase, decrease, concavity
5 integrals
1 area between the curves
1 volume of rotation
7 short-answer or true false questions related to
differentiability, continuity, integrability
(no proofs, no parametric equations, no exponential or logarithm
derivatives, no summation notation)

Limits
When possible, write $\infty$ or $-\infty$ rather than “d.n.e.”.

- $\lim_{x \to 4} \frac{x+4}{x-4} = ?$ Ans. $= \frac{4+4}{4-4} = \frac{0}{0} = 0$
- $\lim_{x \to 4^+} \frac{x+4}{x-4} = ?$ Ans. $= \frac{4+4}{4^+ - 4} = \frac{8}{0^+} = \infty$
- $\lim_{x \to 4^-} \frac{x+4}{x-4} = ?$ Ans. $= \frac{4+4}{4^- - 4} = \frac{8}{0^-} = -\infty$
- $\lim_{x \to -5} \frac{x+5}{x^2+3x-10} = ?$
  Ans. $\lim_{x \to -5} \frac{x+5}{(x+5)(x-2)} = \lim_{x \to -5} \frac{1}{x-2} = -\frac{1}{7}$
- $\lim_{x \to 1} \frac{x-1}{\sqrt{x^2+3x-2}} = ?$
  Ans. $= \lim_{x \to 1} \frac{x-1}{\sqrt{x^2+3x} + 2} \cdot \frac{\sqrt{x^2+3x} - 2}{\sqrt{x^2+3x} - 2}$
  $= \lim_{x \to 1} \frac{(x-1)(\sqrt{x^2+3x} + 2)}{(x^2+3x) - 4}$
  $= \lim_{x \to 1} \frac{(x-1)(\sqrt{x^2+3x} + 2)}{(x-1)(x+4)}$
  $= \lim_{x \to 1} \frac{\sqrt{x^2+3x} + 2}{x+4}$
  $= \frac{\sqrt{1+3(1)} + 2}{1+4} = \frac{4}{5}$
- $\lim_{h \to 0} \frac{3h}{\tan(h)} = ?$
  Ans. $\lim_{h \to 0} \frac{3h}{\sin(h)\cos(h)} = \lim_{h \to 0} \frac{3\cos(h)}{1} \cdot \frac{h}{\sin(h)}$
  $= \lim_{h \to 0} \frac{3\cos(0)}{1} \cdot \frac{h}{\sin(h)} = \frac{3}{1} \cdot 1 = 3$

Derivatives
$f(x) = \frac{1}{x}$ Show that $f'(x) = -\frac{1}{x^2}$ using the definition of derivative.

Ans. $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$
  $= \lim_{h \to 0} \frac{\frac{x-x}{hx(x+h)}}{h(x+h)} = \lim_{h \to 0} \frac{-h}{h(x+h)x} = \lim_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$
- $\frac{1}{(1-x^2)^5} - x^5' = ?$
  Ans. $= \left\{ [(1-x^2)^5 - x^5]^{-1} \right\}'$
  $= \{(1-x^2)^5 - x^5\}^{-2} \cdot \{(1-x^2)^5 - x^5\}'$
  $= \{-1\} \cdot \{(1-x^2)^5 - x^5\}^{-2} \cdot [5(1-x^2)^4(-2x) - 5x^4]$
- $r = 1 - \cos^2(\sqrt{\theta}), \frac{dr}{d\theta} = ?$
  Ans. $\frac{dr}{d\theta} = -2 \cos(\sqrt{\theta})(-\sin(\sqrt{\theta})) \frac{1}{2\sqrt{\theta}} = \cos(\sqrt{\theta}) \sin(\sqrt{\theta}) \frac{1}{\sqrt{\theta}}$
  $r = \cot t, \frac{dr}{dt} = ?$
  Ans. $\frac{dr}{dt} = \left(\frac{1}{\tan t}\right)' = \frac{-(\tan t)'}{\tan^2 t} = -\sec^2 t / \tan^2 t$
  $= -\frac{1}{\cos^2 t} \frac{\cos^2 t}{\sin^2 t} = -\frac{1}{\sin^2 t} = -\csc^2 t$
Harmonic motion, oscillating spring

A 1 kilogram weight hangs on a spring. At rest it has height 0. Its height \( y(t) \) oscillates between heights -1 and 1. When the object is at height 1, the spring is pushing down, when the object is at height -1, the spring is pulling up. At any time, the force of the spring on the object is \( F(t) = -\sin(t) \). At time \( t = 0 \) the object is at height 1 and has velocity 0. Find the position \( y(t) \) at any time \( t \).

\[ \text{Ans.} \]

\( y(t) \) is its position at time \( t \).

Let \( v(t) \) be its velocity and \( a(t) \) its acceleration.

By Newton's law, \( F(t) = ma(t) \), hence its acceleration is

\[ a(t) = \frac{F(t)}{m} = -\sin(t) = -\sin t \]

Given: \( a(t) = -\sin t, \quad v(0) = 0, \quad y(0) = 1 \)

Since \( a(t) = v'(t) \) and \( v(0) = 0 \),

\[ v(t) = \int_0^t v'(x) \, dx + 0 = \int_0^t a(x) \, dx + 0 = \int_0^t -\sin x \, dx + 0 \]

\[ = [-\cos u]_0^t = [\cos u]'_0^t = \cos t - \cos 0 = \cos t - 1 \]

Since \( v(t) = y'(t) \) and \( y(0) = 1 \),

\[ y(t) = \int_0^t y'(x) \, dx + 1 = \int_0^t v(x) \, dx + 1 \]

\[ = \int_0^t (\cos x - 1) \, dx + 1 = [\sin x]'_0^t - [x]'_0^t + 1 \]

\[ = [\sin t - \sin 0] - [t - 0] + 1 = \sin t - t + 1 \]

Implicit differentiation, Tangent equation

\( xy^3 - y = 0 \) Find the equation for the tangent at \((1,1)\).

\[ \text{Ans.} \]

\[ [(1)y^3 + x(3y^2y')] - y' = 0 \]

\[ y^3 + 3xy^2y' - y' = 0 \]

\[ 3xy^2y' - y' = -y^3 \]

\[ (3y^2 - 1)y' = -y^3 \]

\[ y' = \frac{-y^3}{3y^2 - 1} \]

Slope at \((1,1)\): \( m = y'|_{x=1, y=1} = \frac{-1^3}{3 - 1} = \frac{-1}{2} \)

Tangent Equation: \( y - y_0 = m(x - x_0) \)

\[ y - 1 = \frac{-1}{2}(x - 1) \]

\[ y - 1 = \frac{-1}{2}x + \frac{1}{2} \]

Normal Equation: \( y - y_0 = \frac{1}{m}(x - x_0) \)

\[ y - 1 = -3(x - 1) \]

\[ y - 1 = -3x + 3 \]

\[ y = -3x + 4 \]

Related rates

A 5ft. tall boy walks away from a 11 ft. high street lamp at 200 ft/min. He casts a shadow on the ground which starts at his feet. How fast is the shadow lengthening?

\[ \text{Ans.} \]

\[ \frac{ds}{dt} \]

\[ \text{Given:} \quad \frac{dx}{dt} = 200, \quad \frac{x + s}{11} = \frac{s}{5} \]

\[ \text{Eq.:} \quad 5(x + s) = 11s \]

\[ 5x + 5s = 11s, \quad 5x = 6s \]

\[ \text{Diff.:} \quad 5 \frac{dx}{dt} = 6 \frac{ds}{dt}, \]

\[ \text{Ans.:} \quad \frac{ds}{dt} = \frac{5}{6} \frac{dx}{dt} = \frac{5}{6} \cdot 200 = \frac{5}{6} \cdot 100 = \frac{500}{3} \quad \text{ft/min} \]
A balloon is drifting at a constant speed and constant altitude of 800 ft. An observer below the flight path of the balloon measures its angle of elevation \( \theta \). If \( \theta \) is increasing at 1 radian per minute, how fast is the balloon traveling when \( \theta = \frac{\pi}{4} \)?

**Ans.**

**Picture/variables**

Let \( x \) be the distance pictured.

**Want:** \( \frac{dx}{dt} \)

**Given:** \( \frac{d\theta}{dt} = \frac{1}{\text{min.}}, \tan \theta = \frac{800}{x} \)

**Eq.:** \( x = \frac{800}{\tan \theta} \)

**Diff.:** \( \frac{dx}{dt} = 800 \frac{-1}{\tan^2 \theta} \frac{d\theta}{dt} = -800 \frac{\sec^2 \theta}{\tan^2 \theta} \frac{d\theta}{dt} \),

**Ans.:** \( \frac{dx}{dt} = \frac{800}{\cos^2 \frac{\pi}{4}} \frac{1}{\tan^2 \frac{\pi}{4}} \frac{1}{\text{min.}} = \frac{800}{1} = 1600 \)

**Graphs**

- Given the graph of \( f(x) \), graph \( \int_0^x f(t)dt \)

**Graph**

- **Graph** \( y = \frac{x(x-1)}{(x+1)^2} \): On the graph, give the coordinates of critical points and inflection points. Classify as specifically as possible: loc. abs. max/min, loc. max/min., crit. pt., infl. pt. List the intervals of increase/decrease and the intervals of concavity.

**Ans.**

\( y = \frac{x(x-1)}{(x+1)^2}, \quad y' = \frac{3x-1}{(x+1)^3}, \quad y'' = \frac{-6(x-1)}{(x+1)^4} \).

**Roots:** \( x = 0, 1 \)

**Lead term** \( \frac{x(x-1)}{(x+1)^2} \to \frac{x}{(x)^2} = 1 \)

**Horizontal asymptote:** \( y = 1 \)

**Vertical asymptote:** \( x = -1 \)

**Lead term** \( y' = \frac{3x-1}{(x+1)^3} \to \frac{3x}{(x)^3} = \frac{3}{x^2} \)

**Critical points:** \( x = 1/3 \) but not \( x = -1 \) which is undefined.

**Intervals of increase/decrease:**

- up \((-\infty, -1)\), down \((-1, 1/3)\), up \([1/3, \infty)\).

**Local and absolute minimum:** \( y(1/3) = -1/8 \)

**Inflection point:** \( (1, 0) \)

**Lead term** \( y'' = \frac{-6(x-1)}{(x+1)^4} \to \frac{-6x}{(x)^4} = \frac{-6}{x^3} \)

**Intervals of concavity:**

- concave up \((-\infty, -1), (-1, 1), \) concave down \((1, \infty)\).

**Graph:**

Given the graph (red graph) of \( f \), draw the graph of \( f' \).

**Ans.** The graph of \( f' \) is the green graph pictured.
Min-max problem

- A plot consists of a rectangle with an adjoining semicircle on top. You have 80 feet of fencing which must fence the outer perimeter (but not the line which separates the rectangle from the semicircle). Find the radius of the semicircle which maximizes the area of the plot.

**Picture:** Draw the picture and indicate the variables.

\[
\text{length of fence } = 80
\]

\[A = \text{plot area}\]

**Ans.**

**Given:** List the given facts.

Upper semicircle length = \((2\pi r)/2 = \pi r\)

Upper semicircle area = \(\pi r^2/2\)

The lower rectangle perimeter = \(2r + 2y\)

The total fence length is \(\pi r + 2r + 2y\)

Hence the given facts are:

\[\pi r + 2r + 2y = 80\]

\[A = (2r)y + \pi r^2/2\]

**One variable:** Write the variable to be maximized/minimized as a function of one other variable.

\[2y = 80 - 2r - \pi r\]

\[A = (2y) + \pi r^2/2 = r(80 - 2r - \pi r) + \pi r^2/2\]

\[A = 80r - 2r^2 - \pi r^2 + \pi r^2/2 = 80r - (2 + \pi/2)r^2\]

**Critical points:** Differentiate; set the derivative to zero; solve.

\[\frac{dA}{dr} = 80 - (4 + \pi)r\]

\[\frac{dA}{dr} = 0 \iff (4 + \pi)r = 80 \iff r = 80/(4 + \pi)\]

**Answer:** The area is max when the radius is \(80/(4 + \pi)\) feet.

- Find the \(x\) which maximizes the value of \(y\) if \(x^2 - xy + 2y^2 = 1\).

**Ans.** We would like to write \(y\) as a function of \(x\), then find \(y'\), then set \(y' = 0\). But this equation is hard to solve for \(y\), so we will find \(y'\) using implicit differentiation.

\[2x - (1y + xy') + 4yy' = 0\]

\[(4y - x)y' = -2x\]

\[y' = \frac{y - 2x}{4y - x}, \quad y' = 0 \iff y = 2x\]

\[x^2 - x(2x) + 2(2x)^2 = 1, \quad x^2 - 2x^2 + 8x^2 = 1, \quad 7x^2 = 1\]

\[x^2 = 1/7, \quad x = \pm 1/\sqrt{7}, \quad y = 2x \text{ is max when } x = 1/\sqrt{7}\]

- Cylinder is inscribed in a cone of height \(H\) and radius \(R\). Find the height \(h\) and radius \(r\) of the cylinder of maximal volume.

**Given:** \(V = \pi r^2 h, \quad \frac{h}{R - r} = \frac{H}{R}\)

**One variable:** \(V = \pi r^2 H (R - r) = \pi H (Rr^2 - r^3)\)

**Domain:** \(r \in [0, R]\)

**Derivative:** \(V' = \pi H (2Rr - 3r^2)\)

**Crit. Pts:** \(3r = 2R, \quad r = \frac{2}{3}R, \quad r = 0, R\)

**Answer:** \(r = \frac{2}{3}R, \quad h = H \left( R - \frac{2}{3}R \right) = \frac{1}{3}H\)

**Proof.** \(V_0 = V_1 = 0, V_{12\pi/3} > 0\)

- Find the \(x\) which maximizes the value of \(y\) if \(x^2 - xy + 2y^2 = 1\).

**Ans.**

**Given:** \(\sqrt{49} - \sqrt{48} = \sqrt{1}\)

**Linear Approximation.**

**Approximate** \(\sqrt{49} - \sqrt{48} = 1\) with linear approximation.

\[f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}\]

**f'(x)h \approx f(x + h) - f(x) and f(x + h) \approx f(x) + f'(x)h**

\[f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}}, \quad x + h = 48, \quad x = 49, \quad h = 1\]

\[f(x + h) \approx f(x) + f'(x)h \quad \text{becomes}\]

\[\sqrt{49} \approx \sqrt{49} + \frac{1}{2\sqrt{49}}(-1) = 7 - \frac{1}{14} = 7 - \frac{1}{14}\]

\[\sqrt{49} - \sqrt{48} \approx 7 - \frac{1}{14} = \frac{1}{14}\]

**Integrals**

\[\int_0^1 \frac{1}{\sqrt{u}(1 + \sqrt{u})^5} du = ?\]

**Ans.**

\[w = 1 + \sqrt{u}, \quad dw = \frac{1}{2\sqrt{u}}du, \quad 2dw = \frac{du}{\sqrt{u}}, \quad w = \int_{w_0}^{w_1} \frac{1}{w^5}dw = 2 \int_1^2 \frac{w^5}{w} dw = 2 \int_1^2 w^{-5} dw = 2 \left[ \frac{w^{-4}}{-4} \right]_1^2 = \frac{1}{2} \left[ \frac{1}{2^4} - \frac{1}{1^4} \right] = -\frac{1}{2} \left[ \frac{1}{16} - 1 \right] = -\frac{1}{2} \cdot \frac{15}{16} = \frac{15}{32}\]
Areas

- Find the total area (geometric area, not signed area) enclosed by \( y = 2x - x^3 \) and \( y = x^3 \).

\[ \text{Ans.} \]

**Intersections:** \( 2x - x^3 = x^3 \) iff \( 2x - 2x^3 = 0 \) \iff 2x(1 - x^2) = 0 \iff x = -1, 0, 1 \\

**Area:** \( \int_{-1}^{0} (2x - x^3) - x^3 \, dx \) + \( \int_{0}^{1} (2x - x^3) - x^3 \, dx \)

Let's do the indefinite integral first to get an antiderivative (drop the +C from the indefinite integral),

\[
\int 2x - x^3 - x^3 \, dx = \int 2x - 2x^3 \, dx
\]

\[
= 2 \int x - x^3 \, dx = 2\left[ \frac{x^2}{2} - \frac{x^4}{4} \right] = x^2 - \frac{1}{2}x^4 + C \quad \text{Delete C.}
\]

Hence the area

\[
= [(x^2 - \frac{1}{2}x^4)]_0^1 + [(x^2 - \frac{1}{2}x^4)]_0^1
\]

\[
= \left[ (1 - 1) - (0 - 0) \right]_0^1 + \left[ (1 - 1) - (0 - 0) \right]_0^1
\]

\[
= 1 - \left( \frac{1}{2} \right) + 1 = \frac{1}{2} + \frac{1}{2} = 1
\]

U-substitution

\[
\int \frac{2}{x^3} \cos^2\left( \frac{1}{x^2} \right) \, dx = ?
\]

\[ \text{Ans.} \]

u = 1/x^2, \ du = -2/x^3, \ \frac{-du}{2} = \frac{1}{x^3} \, dx

\[
= \int 2 \cos^2(u) \frac{-du}{2} = -\int \cos^2(u) \, du
\]

\[
= -\left( \frac{1}{2}u + \frac{1}{4} \sin(2u) \right) = -\frac{1}{2}x^2 - \frac{1}{4} \sin \frac{2}{x^2} + C
\]

\[
\int_{-1}^{0} 15x^5 \sqrt{x^3 + 1} \, dx = ?
\]

\[ \text{Ans.} \]

\[
= 15 \int_{-1}^{0} \sqrt{x^3 + 1} \, x^5 \, dx
\]

\[
u = x^3 + 1, \ du = 3x^2 \, dx, \ \frac{du}{3} = x^2 \, dx, \ x^3 = u - 1
\]

\[
= 15 \int_{-1}^{0} \sqrt{u-1} \, (u-1) \sqrt{u} \, \frac{du}{3}
\]

\[
= \frac{15}{3} \int_{0}^{1} u^{3/2} - u^{1/2} \, du = 5\left[ \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} \right]_0^1
\]

\[
= 5\left[ \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_0^1 = 5\left[ \frac{2}{5} \left( \frac{2}{3} \right) + (0 - 0) \right] = 2 - \frac{10}{3} = \frac{4}{3}
\]

Riemann sums

- Estimate \( \int_{0}^{1} x^3 \, dx \) with a Riemann sum with 3 equal-length intervals where the height of each rectangle is value of the function in the middle of the interval.

\[ \text{Ans.} \]

\[
= \left( \frac{3}{2} \right)^3 1 + \left( \frac{5}{2} \right)^3 1 + \left( \frac{7}{2} \right)^3 1
\]

Volumes

- The curves \( y = x^2 \) and \( y = x^{1/3} \) enclose a region above \([0, 1]\).

(a) Rotate the region around the x-axis. Find the volume.

(b) Rotate the region around the y-axis. Find the volume.

\[ \text{Ans.} \]

(a) Find the volume using washers.

\[ r_{outer}(x) = x^{1/3}, \ r_{inner}(x) = x^2 \]

\[
\text{volume} = \pi \int_{0}^{1} (r_{outer}(x))^2 - (r_{inner}(x))^2 \, dx
\]

\[
= \pi \int_{0}^{1} x^{2/3} - x^4 \, dx = \pi \left[ \frac{x^{5/3}}{5/3} - \frac{x^5}{5} \right]_{0}^{1}
\]

\[
= \pi \left[ \left( \frac{5}{3} \right) - \left( \frac{5}{3} \right) \right] = \pi \left( \frac{3}{5} \right) - \frac{1}{5} = \frac{2\pi}{5}
\]

(b) Find the volume using shells.

\[ r(x) = x, \ h(x) = x^{1/3} - x^2 \]

\[
\text{volume} = 2\pi \int_{0}^{1} 2\pi rh \, dx = 2\pi \int_{0}^{1} x(x^{1/3} - x^2) \, dx
\]

\[
= 2\pi \int_{0}^{1} (x^{4/3} - x^3) \, dx = 2\pi \left[ \frac{x^{7/3}}{7/3} - \frac{x^4}{4} \right]_{0}^{1}
\]

\[
= 2\pi \left[ \left( \frac{7}{3} \right) - \frac{1}{4} \right] = 2\pi \left( \frac{3}{7} - \frac{1}{4} \right) = \pi \left( \frac{6}{7} - \frac{1}{4} \right)
\]

\[
= \pi \left( \frac{12}{14} - \frac{7}{14} \right) = \frac{5\pi}{14}
\]