UNIFIED ANALYSIS OF FINITE ELEMENT METHODS FOR PROBLEMS WITH MOVING BOUNDARIES*

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Abstract. We present a unified analysis of finite element methods for problems with prescribed moving boundaries. In particular, we study an abstract parabolic problem posed on a moving domain with prescribed evolution, discretized in space with a finite element space that is associated with a moving mesh that conforms to the domain at all times. The moving mesh is assumed to evolve smoothly in time, except perhaps at a finite number of remeshing times where the solution is transferred between finite element spaces via a projection.

A key result of our analysis is an abstract estimate for the L^2-norm of the error between the exact and semidiscrete solutions at a fixed positive time, expressed in terms of the total variation in time of a quantity that measures the difference between the exact solution at time t and its elliptic projection onto the finite element space at time t. Specializing the abstract estimate to particular choices of the mesh motion strategy, finite element space, and projector leads to error estimates in terms of the mesh spacing for various semidiscrete schemes. In particular, the estimate can be specialized to conventional arbitrary Lagrangian-Eulerian (ALE) schemes with remeshing as well as schemes based upon universal meshes, where the mesh motion is derived from small deformations of a periodically updated reference subtriangulation of a background mesh that contains the moving domain. We demonstrate such an application by deducing error estimates of optimal order in the mesh spacing for ALE schemes under mild assumptions on the nature of the mesh deformation and the regularity of the exact solution and the moving domain.

Key words. Moving-boundary problem, ALE, a priori error analysis, dynamic mesh

AMS subject classifications. 65M15, 65M60, 35R37, 35K90

1. Introduction. Problems with moving boundaries are ubiquitous in science and engineering yet challenging to solve numerically. A common approach for solving such problems using finite elements is to choose a conforming mesh of the domain at time $t = 0$ and to prescribe a deformation of that mesh to discretize the domain at times $t > 0$, remeshing as often as needed to maintain a mesh of adequate quality. On this moving mesh, a numerical solution is generated by solving a finite element spatial discretization of the governing equations over the intervals of time for which the mesh deforms smoothly. At the instants at which remeshing occurs, the numerical solution is transferred from one finite element space to another via a projection, such as interpolation or the L^2-projection. We refer to such methods as *deforming-mesh methods* in this paper. The aim of this paper is to carry out a unified a priori error analysis for such methods in the spatially discrete, temporally continuous setting.

In a typical deforming-mesh method, the deformation of the mesh is constructed by solving a system of equations, such as those of linear elasticity, for the nodal positions [24, 15, 25], or, for instance, by using mesh optimization, mesh morphing, or mesh smoothing techniques [27, 35, 26, 28, 34]. For many strategies, the number of instants at which remeshing is performed during the course of a given simulation remains bounded as the spatial discretization is refined. Methods with these characteristics, commonly known as arbitrary Lagrangian-Eulerian (ALE) schemes, are not the only deforming-mesh methods, however. Schemes based upon *universal meshes* [21, 20, 32]

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can likewise be viewed as deforming-mesh methods. In these schemes, the mesh motion is derived from small deformations of a periodically updated reference subtriangulation of a background mesh that contains the moving domain for all times. The mesh deformations deviate from the identity only in a band of elements near the moving boundary, and the number of instants at which "remeshing" (updating the reference subtriangulation) is performed during the course of a given simulation grows unboundedly under mesh refinement.

An important feature of our analysis is that it applies to both of the settings described above, even though the ultimate convergence orders of the two approaches with respect to the mesh spacing differ markedly. We accomplish this in a unified manner by leaving the precise choice of the mesh motion strategy, remeshing times, finite elements, and projector unspecified throughout much of the analysis.

The analysis of deforming-mesh methods has received the attention of several prior authors, though none to our knowledge have adopted the same focus or scope as the present work. Many efforts have addressed the stability of temporal discretizations [17, 18, 3], often focusing on a well-known condition (the so-called *Geometric Conservation Law*) that ensures stability of certain low-order schemes [23, 16]. Bonito and co-authors [5, 4] study ALE schemes in the temporally discrete, spatially continuous setting, and they present a family of high-order time integrators that achieve optimal order of accuracy in time for a model parabolic problem on a moving domain. They derive, among other things, sufficient conditions to ensure mesh-motion-independent stability of the time integrators. Gastaldi [19] proves a priori error estimates for a second-order accurate fully discrete scheme, and the spatially discrete analysis presented therein bears some similarity to the present work. Our analysis, however, generalizes Gastaldi's in several key respects. We consider general mesh deformations, rather than those derived from solutions to the equations of linear elasticity; we account for remeshing; we consider finite element spaces of arbitrary order, rather than piecewise linears; and we allow for arbitrary elements (simplicial, hexahedral, curved, etc.) to compose the mesh. Another study that is much in the spirit of the present work is Elliott & Venkataraman's analysis [12] of a finite element method for an advection-diffusion equation on an evolving surface, which considers the use of piecewise linears without remeshing. Finally, Dupont [11] analyzes finite element methods on moving meshes over fixed domains and accounts for remeshing. There, the focus is on a special choice of norm over the spacetime domain in which the error is quasi-optimal.

The primary contributions of this paper are presented in two theorems and one corollary. The first is Theorem 3.1, which provides an abstract upper bound on the L^2-norm of the error between the exact and semidiscrete solutions at a fixed positive time. This bound is expressed in terms of the total variation in time of a quantity ρ that measures difference between the exact solution at time t and its elliptic projection onto the finite element space at time t. The second contribution is Theorem 3.2, which bounds the material time derivative of ρ. Finally, Corollary 3.3 illustrates an application of Theorems 3.1 and 3.2. It states an error estimate of optimal order in the mesh spacing for ALE schemes under mild assumptions on the nature of the mesh deformation and the regularity of the exact solution and the moving domain.

The assumptions on the mesh deformation stipulated in Corollary 3.3, particularly (3.11), are a central result of the analysis, since they (together with more basic assumptions on the finite element spaces, exact solution, and moving domain) provide sufficient conditions for a mesh motion strategy to deliver a discretization with an optimally convergent solution. These conditions, which effectively place restrictions on the velocity of the mesh, supplement standard conditions on the mesh (such as shape regularity) that ensure optimal approximation properties of the finite element spaces at each instant in time. For practitioners, checking (3.11) amounts to calculating norms of a certain bilinear form related to the mesh velocity; see Section 6.1 for an example.

The level of generality adopted in our analysis implies that the abstract error estimates apply regardless of the manner and frequency with which remeshing is performed, and regardless of the metrics for mesh quality that are used to guide remeshing. Of course, the quality of the mesh at each instant in time influences the ultimate error estimates via the approximation properties of the corresponding finite element spaces.

The use of the elliptic projection in the a priori error analysis of finite element methods for parabolic problems on fixed domains is a well-established technique used heavily in the text of Thomee [36]. Our analysis is, to some extent, a generalization of this strategy to the setting in which the domain is time-dependent and the finite element spaces are permitted to change abruptly at a finite number of instants.

This paper is organized as follows. In Section 2, we state an abstract parabolic problem on a moving domain and delineate a class of numerical methods to be analyzed. In Section 3, we present the statements of Theorem 3.1, Theorem 3.2, and Corollary 3.3. We prove Theorem 3.1 in Section 4, and we prove Theorem 3.2 and Corollary 3.3 in Section 5. In Section 6, we check the hypotheses of Corollary 3.3 when the PDE under consideration is the diffusion equation and the mesh motion under consideration satisfies certain uniform bounds on its velocity. Some concluding remarks are given in Section 7.

2. Continuous Problem and its Discretization.

This section details the moving-boundary problem under consideration and the class of numerical methods to be analyzed. The problem we consider, stated in an abstract form, encompasses a range of parabolic partial differential equations posed on moving domains with prescribed evolution, including (diffusion-dominated) convection-diffusion-reaction equations.

We begin by stating the continuous problem and its weak formulation in Section 2.1. We discuss its spatial discretization via finite elements in Section 2.2. Finally, in Section 2.3, we present the class of numerical methods under scrutiny.

2.1. Continuous Problem.

This paper considers a moving-boundary problem posed on an evolving domain $\Omega^t \subset \mathbb{R}^d$, $t \in [0,T]$, $d \geq 1$, where Ω^t is open, bounded, and Lipschitz for every $t \in [0,T]$, and T is a fixed positive number. We denote by $\Omega \subset \mathbb{R}^{d+1}$ the spacetime domain

$$\Omega = \{(x,t) \in \mathbb{R}^{d+1} \mid x \in \Omega^t, 0 < t < T\}.$$

To state precisely the moving-boundary problem under consideration, we require the following notation and definitions.

Notation. For $s \geq 0$, $1 \leq p \leq \infty$, and $D = \Omega^t$ or $D = \Omega$, we denote by $W^{s,p}(D)$ the Sobolev space of differentiability s and integrability p, equipped with the norm $\|\cdot\|_{s,p,D}$ and semi-norm $|\cdot|_{s,p,D}$. We denote $H^s(D) = W^{s,2}(D)$ for every $s \geq 0$ and $L^p(D) = W^{0,p}(D)$ for every $1 \leq p \leq \infty$. We use $H^1_0(D)$ to denote the space of functions in $H^1(D)$ with vanishing trace on ∂D, and we denote by $H^{-1}(D)$ its dual. We use $W^{s,p}(D)^d$, $L^p(D)^d$, $H^s(D)^d$, and $H^{-1}(D)^d$ to denote the analogous spaces of vector valued functions $u : D \to \mathbb{R}^d$.

For a given Banach space B and integer $s \geq 0$, we denote the Bochner spaces

$$W^{s,p}(0,T;B) = \left\{ U : (0,T) \to B \mid \int_0^T \sum_{i=0}^s \|U^{(i)}(t)\|_B^p \, dt < \infty \right\}, \quad 1 \leq p < \infty,$$
$$W^{s,\infty}(0,T;B) = \left\{ U : (0,T) \to B \mid \sup_{0<t<T} \sum_{i=0}^s \|U^{(i)}(t)\|_B < \infty \right\},$$

where $U^{(i)}$ denotes the i^{th} weak time derivative of a Banach space-valued function U. We denote $H^s(0,T;B) = W^{s,2}(0,T;B)$ for every $s \geq 0$ and $L^p(0,T;B) = W^{0,p}(0,T;B)$ for every $1 \leq p \leq \infty$.

For notational convenience, we denote $\mathcal{V}^t = H_0^1(\Omega^t)$, and we define the space

$$\mathcal{U} = \{U \in L^2(0,T;H_0^1(\Omega^0)) \mid U^{(1)} \in L^2(0,T;H^{-1}(\Omega^0))\}.$$

Note that $\mathcal{U} \subset C([0,T], L^2(\Omega^0))$ [14, Chapter 5.9, Theorem 3], the space of continuous functions in $[0,T]$ taking values in $L^2(\Omega^0)$.

Finally, we introduce the notion of a regular domain deformation (see also [5, Definition 2.1]).

Definition 2.1. *For $0 \leq \tau_0 < \tau_1 \leq T$, let $\{\varphi^t : \Omega^{\tau_0} \to \Omega^t \mid \tau_0 < t \leq \tau_1\}$ be a family of continuous maps. We say that $\{\varphi^t \mid \tau_0 < t \leq \tau_1\}$ is a* regular domain deformation *over $(\tau_0, \tau_1]$ if the following conditions are satisfied:*

(2.1.i) The map $t \mapsto \varphi^t$ belongs to $W^{1,\infty}\left(\tau_0, \tau_1; W^{1,\infty}(\Omega^{\tau_0})^d\right)$.

(2.1.ii) For every $t \in (\tau_0, \tau_1]$, φ^t is surjective.

(2.1.iii) There exists $C > 0$ independent of t, X, and Y such that for every $t \in (\tau_0, \tau_1]$ and every $X, Y \in \Omega^{\tau_0}$,

$$\|\varphi^t(X) - \varphi^t(Y)\|_{\mathbb{R}^d} \geq C\|X - Y\|_{\mathbb{R}^d},$$

where $\|\cdot\|_{\mathbb{R}^d}$ denotes the Euclidean distance in $\mathbb{R}^d$.

(2.1.iv) $\varphi^{\tau_0+} := \lim_{t \searrow \tau_0} \varphi^t = i$ in $W^{1,\infty}(\Omega^{\tau_0})^d$, where i denotes the identity and $\lim_{t \searrow \tau_0}$ denotes the limit as t approaches τ_0 from above.

Later in this paper, we reuse the letter C to denote a generic constant, not necessarily the same at each occurrence, whose dependence (or lack thereof) on other parameters of interest will be specified as needed.

Note that if $\{\varphi^t \mid \tau_0 < t \leq \tau_1\}$ is a regular domain deformation, then conditions (2.1.ii-2.1.iii) ensure that for each $t \in (\tau_0, \tau_1]$, φ^t is bijective with Lipschitz inverse. Furthermore, a function $u : \Omega^t \to \mathbb{R}$ belongs to $\mathcal{V}^t = H_0^1(\Omega^t)$ if and only if $u \circ \varphi^t \in \mathcal{V}^{\tau_0} = H_0^1(\Omega^{\tau_0})$; see [5, Section 2.1].

As a regularity requirement on the domain's evolution, we shall assume the existence of a regular domain deformation $\{\psi^t : \Omega^0 \to \Omega^t \mid 0 < t \leq T\}$ for which the map $(X,t) \mapsto \psi^t(X)$ belongs to $C^2(\Omega^0 \times [0,T], \mathbb{R}^d)$. We denote

$$\mathcal{W} = \{u : \Omega \to \mathbb{R} \mid t \mapsto u(\psi^t(\cdot), t) \in \mathcal{U}\},$$
$$\mathcal{F} = \{f : \Omega \to \mathbb{R} \mid t \mapsto f(\psi^t(\cdot), t) \in C([0,T], L^2(\Omega^0))\}.$$

Moving boundary problem. Our interest is in problems of the following form: Given $f \in \mathcal{F}$ and $u^0 \in \mathcal{V}^0$, find $u \in \mathcal{W}$ such that $u(\cdot, 0) = u^0$ and

$$m^t(\dot{u}, w) + a^t(u, w) = m^t(f, w) \quad \forall w \in \mathcal{V}^t \tag{2.1}$$

for a.e. $t \in (0,T)$, where $\dot{u} := \frac{\partial u}{\partial t}$,

$$m^t(u,w) := \int_{\Omega^t} uw \, dx,$$

and $a^t : \mathcal{V}^t \times \mathcal{V}^t \to \mathbb{R}$ is a time-dependent bilinear form satisfying the following hypotheses:

(2.i) There exist positive constants M_0 and α_0 independent of t such that for every $t \in [0,T]$ and every $u, w \in \mathcal{V}^t$,

$$a^t(u,u) \geq \alpha_0 \|u\|^2_{1,2,\Omega^t}, \tag{2.2}$$

$$|a^t(u,w)| \leq M_0 \|u\|_{1,2,\Omega^t} \|w\|_{1,2,\Omega^t}. \tag{2.3}$$

(2.ii) For any regular domain deformation $\{\varphi^t : \Omega^0 \to \Omega^t \mid 0 < t \leq T\}$, there exists $C > 0$ such that for every $U, W \in \mathcal{V}^0$, the map

$$t \mapsto y(t) := a^t(U \circ (\varphi^t)^{-1}, W \circ (\varphi^t)^{-1})$$

is Lipschitz with

$$|y(\tau_1) - y(\tau_0)| \leq C \|U\|_{1,2,\Omega^0} \|W\|_{1,2,\Omega^0} |\tau_1 - \tau_0| \tag{2.4}$$

for every $\tau_0, \tau_1 \in (0,T]$.

We shall assume the existence of a unique solution $u \in \mathcal{W}$ to (2.1) satisfying the additional regularity

$$\int_0^T \|u\|_{2,2,\Omega^t} + \|\dot{u}\|_{1,2,\Omega^t} \, dt < \infty. \tag{2.5}$$

Note that this assumption guarantees that for any regular domain deformation $\{\varphi^t : \Omega^0 \to \Omega^t \mid 0 < t \leq T\}$, the map $t \mapsto u(\varphi^t(\cdot), t)$ belongs to $W^{1,1}(0,T; H^1_0(\Omega^0))$. Furthermore, the embedding $W^{1,1}(0,T; H^1_0(\Omega^0)) \subset C([0,T], H^1_0(\Omega^0))$ [14, Chapter 5.9, Theorem 2] ensures that the trace of u on any constant-time slice Ω^t is a well-defined member of $\mathcal{V}^t$; that is, $u(\cdot, t) \in \mathcal{V}^t$ for every $t \in [0,T]$.

For $0 \leq \tau_0 \leq \tau_1 \leq T$, let $\Phi^{\tau_1}_{\tau_0} : \mathcal{V}^{\tau_0} \to \mathcal{V}^{\tau_1}$ denote the flow of the differential equation (2.1). That is, if $y \in \mathcal{V}^{\tau_0}$ and u solves (2.1) with the initial condition $u(\cdot, \tau_0) = y$, then $\Phi^{\tau_1}_{\tau_0} y = u(\cdot, \tau_1)$.

2.2. Spatial Discretization. In what follows, we present the general form of a finite element spatial discretization of (2.1) obtained via Galerkin projection onto an evolving finite element space. It is assumed that the finite element space is associated with a deforming mesh that conforms to the domain at all times and evolves smoothly in time, except at a finite number of remeshing times where the solution is transferred between finite element spaces via a projection. We use the term *mesh of* Ω^t to refer to a finite collection of compact, connected, Lipschitz sets (*elements*) with non-empty interior that provide a partition of Ω^t. For an element K of a mesh of Ω^t, we denote its diameter by h_K.

We begin by introducing the notion of a deforming mesh, which we allow to evolve in a discontinuous fashion.

Definition 2.2. *We say that* $\{\mathcal{T}^t_h \mid 0 < t \leq T\}$ *is a* deforming mesh with remeshing times $0 = t^0 < t^1 < \cdots < t^N = T$ and mesh spacing h *if:*

(2.2.i) For each $t \in (0,T]$, $\mathcal{T}^t_h$ *is a mesh of* Ω^t.

(2.2.ii) $h = \sup_{0<t\le T} \max_{K\in\mathcal{T}_h^t} h_K$.

(2.2.iii) *For every* $n = 1, 2, \ldots, N$*, there exists a mesh* $\mathcal{T}_h^{t_+^{n-1}}$ *of* $\Omega^{t^{n-1}}$ *and a regular domain deformation* $\{\varphi_h^t : \Omega^{t^{n-1}} \to \Omega^t \mid t^{n-1} < t \le t^n\}$ *such that for every* $t \in (t^{n-1}, t^n]$*,*

$$K \in \mathcal{T}_h^{t_+^{n-1}} \iff \varphi_h^t(K) \in \mathcal{T}_h^t.$$

Note that the bijectivity of φ_h^t in (2.2.iii) excludes the possibility of inverted elements.

For a given deforming mesh $\{\mathcal{T}_h^t\}_t$, we denote by $v_h : \Omega \to \mathbb{R}^d$ the vector field

$$v_h(x,t) = \left.\frac{\partial}{\partial t}\right|_X \varphi_h^t(X), \quad X \in \Omega^{t^{n-1}},\, t \in (t^{n-1}, t^n],$$

where $x = \varphi_h^t(X)$. We refer to v_h as the *mesh velocity* in the sequel. For a function $w : \Omega \to \mathbb{R}$, we denote by $D_t w : \Omega \to \mathbb{R}$ the function

$$D_t w(x,t) = \left.\frac{\partial}{\partial t}\right|_X W(X,t), \quad X \in \Omega^{t^{n-1}},\, t \in (t^{n-1}, t^n]$$

whenever the right-hand side is defined, where $W(\cdot,t) = w(\varphi_h^t(\cdot),t)$ and $x = \varphi_h^t(X)$. We refer to $D_t w$ as the *material time derivative* of w. The chain rule for weakly differentiable functions [37, Theorem 2.2.2] shows that if $w \in W^{1,1}(\Omega)$, then $D_t w$ exists and the well-known relation

$$D_t w = \dot{w} + v_h \cdot \nabla_x w$$

holds.

Finally, we introduce finite element spaces that evolve in concert with $\{\mathcal{T}_h^t\}_t$.

DEFINITION 2.3. *Let* $\{\mathcal{T}_h^t \mid 0 < t \le T\}$ *be a deforming mesh with remeshing times* $0 = t^0 < t^1 < \cdots < t^N = T$*. We say that* $\{\mathcal{V}_h^t \mid 0 < t \le T\}$ *is an* evolving finite element space *over* $\{\mathcal{T}_h^t\}_t$ *if:*

(2.3.i) *For every* $0 < t \le T$*,* $\mathcal{V}_h^t$ *is a finite-dimensional subspace of* $\mathcal{V}^t$*.*

(2.3.ii) *For every* $n = 1, 2, \ldots, N$ *there exist functions* $\{N_a\}_a$ *such that the functions*

$$n_a^t = N_a \circ (\varphi_h^t)^{-1} \tag{2.6}$$

form a basis for $\mathcal{V}_h^t$ *for each* $t \in (t^{n-1}, t^n]$*, where* φ_h^t *is the map described in (2.2.iii).*

We denote $\mathcal{V}_h^{t_+^{n-1}} = \operatorname{span}\{N_a\}_a$ and remark that in general, $\mathcal{V}_h^{t_+^{n-1}} \neq \mathcal{V}_h^{t^{n-1}}$.

Galerkin projection. On each interval $(t^{n-1}, t^n]$, a Galerkin projection of (2.1) onto an evolving finite element space $\{\mathcal{V}_h^t\}_t$ reads: Find $u_h \in \mathcal{V}_h^{(t^{n-1},t^n]}$ such that

$$m^t(\dot{u}_h, w_h) + a^t(u_h, w_h) = m^t(f, w_h) \quad \forall w_h \in \mathcal{V}_h^t \tag{2.7}$$

for every $t \in (t^{n-1}, t^n]$, where

$$\mathcal{V}_h^{(t^{n-1},t^n]} = \left\{ (x,t) \mapsto \sum_a \mathbf{u}_a(t) n_a^t(x) \;\middle|\; \mathbf{u}_a = \bar{\mathbf{u}}_a|_{(t^{n-1},t^n]} \text{ for some } \bar{\mathbf{u}}_a \in C^1([t^{n-1}, t^n]) \right\}.$$

Expanding u_h as a linear combination of the basis functions (2.6) shows that (2.7) is equivalent to a linear system of ordinary differential equations which, via an application of the Cauchy-Lipschitz theorem [1], admits a unique solution $u_h \in \mathcal{V}_h^{(t^{n-1},t^n]}$ for any initial condition $u_h(\cdot, t_+^{n-1}) \in \mathcal{V}_h^{t_+^{n-1}}$.

For $t^{n-1} < \tau_0 \leq \tau_1 \leq t^n$, $1 \leq n \leq N$, let $\Phi_{h,\tau_0}^{\tau_1} : \mathcal{V}_h^{\tau_0} \to \mathcal{V}_h^{\tau_1}$ denote the flow of the differential equation (2.7). That is, if $y_h \in \mathcal{V}_h^{\tau_0}$ and u_h solves (2.7) with the initial condition $u_h(\cdot, \tau_0) = y_h$, then $\Phi_{h,\tau_0}^{\tau_1} y_h = u_h(\cdot, \tau_1)$.

Remark. Trivially, (2.7) is equivalent to the problem: Find $u_h \in \mathcal{V}_h^{(t^{n-1},t^n]}$ such that

$$m^t(D_t u_h, w_h) - b_h^t(u_h, w_h) + a^t(u_h, w_h) = m^t(f, w_h) \quad \forall w_h \in \mathcal{V}_h^t \tag{2.8}$$

for every $t \in (t^{n-1}, t^n]$, where

$$b_h^t(u, w) = \int_{\Omega^t} (v_h \cdot \nabla_x u) w \, dx. \tag{2.9}$$

Of course, it also holds that the solution u to the continuous problem (2.1) satisfies

$$m^t(D_t u, w) - b_h^t(u, w) + a^t(u, w) = m^t(f, w) \quad \forall w \in \mathcal{V}^t \tag{2.10}$$

for a.e. $t \in (t^{n-1}, t^n]$. These formulations are the well-known ALE formulations familiar to ALE practitioners.

2.3. Semidiscrete Evolution. We now fix an evolving finite element space $\{\mathcal{V}_h^t \subset \mathcal{V}^t \mid 0 < t \leq T\}$ with remeshing times $0 = t^0 < t^1 < \dots t^N = T$ and study numerical methods obtained by solving (2.8), (or, equivalently, (2.7)) over each interval $(t^{n-1}, t^n]$ and transferring the solution across remeshing times via a projection. We view $\{\mathcal{V}_h^t\}_t$ as a member of a family of evolving finite element spaces parametrized by $h \in (0, h_0]$, where h_0 is a fixed positive constant. Note that we allow the number N of remeshing times t^n, as well as the values of t^n, to depend on h. When we wish to emphasize this dependence, we write $N(h)$ and $t^n(h)$, respectively; otherwise, we simply write N and t^n (with the dependence on h implied).

Projector onto finite element spaces. To transfer the solution $u_h(\cdot, t^n) \in \mathcal{V}_h^{t^n}$ to $\mathcal{V}_h^{t_+^n}$ at each remeshing time t^n, we assume that a linear projector

$$p_h^{t_+^n} : \mathcal{V}_h^{t^n} + \mathcal{V}_h^{t_+^n} \to \mathcal{V}_h^{t_+^n},$$

is adopted for each $n = 1, 2, \dots, N-1$ and a linear projector $p_h^{0_+} : \mathcal{V}^0 \to \mathcal{V}_h^{0_+}$ is adopted when $n = 0$.

We make the following hypotheses on the projectors:

(3.1.i) The projectors are surjective; equivalently, $p_h^{t_+^n}\Big|_{\mathcal{V}_h^{t_+^n}} = i$ for every $h \leq h_0$ and every $n = 0, 1, \dots, N-1$, where i denotes the identity.

(3.1.ii) There exists a constant C_p independent of h such that for every $h \leq h_0$ and every $n = 0, 1, \dots, N-1$, the inequality

$$\|p_h^{t_+^n} w\|_{0,2,\Omega^{t^n}} \leq C_p \|w\|_{0,2,\Omega^{t^n}}$$

holds for every $w \in \mathcal{V}_h^{t^n} + \mathcal{V}_h^{t_+^n}$ (if $n > 0$) and every $w \in \mathcal{V}^0$ (if $n = 0$).

Recursions. For $n = 1, 2, \dots, N$, we denote by

$$\mathfrak{f}^n : \mathcal{V}^{t^{n-1}} \to \mathcal{V}^{t^n}$$

the advancement of the solution to (2.10) (or, equivalently, (2.1)) from $t = t^{n-1}$ to $t = t^n$, i.e. $\mathfrak{f}^n = \Phi_{t^{n-1}}^{t^n}$, and by

$$\mathfrak{f}_h^n : \mathcal{V}_h^{t_+^{n-1}} \to \mathcal{V}_h^{t^n}$$

the advancement of the semidiscrete solution to (2.8), (or, equivalently, (2.7)) from $t = t_+^{n-1}$ to $t = t^n$, i.e. $\mathfrak{f}_h^n = \Phi_{h,t_+^{n-1}}^{t^n}$.

In terms of the operators defined above, the values of the exact solution $u^n := u(\cdot, t^n)$ at the temporal nodes satisfy the recursion

$$u^n = \mathfrak{f}^n u^{n-1} \tag{2.11}$$

with the initial condition

$$u^0 = u(\cdot, 0). \tag{2.12}$$

In this paper, we study numerical approximations $u_h^n \approx u^n$ generated by recursions of the form

$$u_h^n = \mathfrak{f}_h^n p_h^{t_+^{n-1}} u_h^{n-1} \tag{2.13}$$

with the initial condition

$$u_h^0 = u(\cdot, 0). \tag{2.14}$$

Note that $u_h^n \in \mathcal{V}_h^{t^n}$ for every $n \geq 1$. In contrast, $u_h^0 \in \mathcal{V}^0$ (though the numerical algorithm immediately projects u_h^0 onto $\mathcal{V}_h^{0+}$).

3. Statement of Results.

3.1. Abstract Error Estimate. We now present an abstract estimate for the global error

$$\varepsilon^n = u^n - u_h^n$$

at $n = N$.

A statement of the estimate makes use of an *elliptic projector*

$$r_h^t : \mathcal{V}^t \to \mathcal{V}_h^t$$

associated with the bilinear form $a^t(u, w) - b_h^t(u, w)$, where b_h^t is given by (2.9). Since this bilinear form is not necessarily coercive, it is useful to consider a modified bilinear form

$$a_h^t(u, w) = a^t(u, w) - b_h^t(u, w) + \kappa m^t(u, w), \tag{3.1}$$

with $\kappa \geq 0$ chosen such that a_h^t is coercive, uniformly in t and h. This is accomplished in Section 4.3, under the following assumption on the mesh velocity:

(3.1.iii) There exists v_{max} independent of h such that

$$\|v_h\|_{0,\infty,\Omega} \leq v_{max}$$

for every $h \leq h_0$.

With this assumption, a suitable choice for κ is

$$\kappa = \frac{v_{max}^2}{2\alpha_0}, \tag{3.2}$$

where α_0 is the constant of coercivity of a^t assumed in (2.i); see Lemma 4.3 below.

Now define r_h^t for $t \in (0, T]$ via

$$a_h^t(r_h^t u - u, w_h) = 0 \quad \forall w_h \in \mathcal{V}_h^t. \tag{3.3}$$

Let

$$\rho(t) = e^{-\kappa t}(r_h^t u(\cdot, t) - u(\cdot, t)) \tag{3.4}$$

for $t \in (0, T]$. To simplify the forthcoming analysis, it is convenient to also define $\rho(0) = 0$ and $\rho(T_+) = 0$.

The following theorem will be proved in Section 4.

THEOREM 3.1. *Let $\{u_h^n\}_{n=0}^N$ be generated by the recursion (2.13) with the initial condition (2.14), using an evolving finite element space $\{\mathcal{V}_h^t \mid 0 < t \leq T\}$ satisfying (3.1.iii) and projectors $\{p_h^{t_+^n}\}_{n=0}^{N-1}$ satisfying (3.1.i-3.1.ii). Then for every $h \leq h_0$, the error $\varepsilon^N = u(\cdot, T) - u_h^N$ satisfies*

$$\begin{aligned}\|\varepsilon^N\|_{0,2,\Omega^T} \leq & \sum_{n=1}^{N} C_p^{N-n} \Big(\int_{t^{n-1}}^{t^n} e^{\kappa t} \|D_t \rho\|_{0,2,\Omega^t}\, dt + e^{\kappa t^n} \|\rho(t_+^n) - \rho(t^n)\|_{0,2,\Omega^{t^n}} \Big) \\ & + C_p^N \|\rho(0_+)\|_{0,2,\Omega^0}\end{aligned}$$

with ρ given by (3.4) and κ given by (3.2).

The preceding theorem reveals that a study of the error at time T reduces to an analysis of ρ, the (scaled) difference between the exact solution $u(\cdot, t)$ at times $t \in (0, T]$ and its elliptic projection onto the current finite element space. The error bound resembles the total variation of ρ: a (weighted) time integral of the norm of $D_t\rho$, plus a (weighted) summation of the jumps in ρ across the times of remeshing, where the weights are related to the constant κ and the stability constant C_p of the projector. The time integral of $\|D_t\rho\|_{0,2,\Omega^t}$ encapsulates the error introduced by Galerkin projection of the governing equations, while the jumps in ρ encapsulate the errors introduced by projecting onto a new finite element space at each remeshing time t^n.

3.2. Bound on $D_t\rho$. Our second theorem provides an upper bound for the L^2-norm of the material time derivative of $\rho(t) = e^{-\kappa t}(r_h^t u(\cdot, t) - u(\cdot, t))$. It will require the following additional hypotheses concerning elliptic regularity, the approximation properties of $\mathcal{V}_h^t$, and the mesh velocity v_h.

(3.2.i) There exists $C > 0$ independent of h and t such that for every $0 < t \leq T$, every $h \leq h_0$, and every $f \in L^2(\Omega^t)$, the solution $y \in \mathcal{V}^t$ to the problem

$$a_h^t(w, y) = m^t(f, w) \quad \forall w \in \mathcal{V}^t$$

satisfies

$$\|y\|_{2,2,\Omega^t} \leq C \|f\|_{0,2,\Omega^t}. \tag{3.5}$$

(3.2.ii) There exists $C > 0$ independent of h and t such that for every $0 < t \leq T$, every $h \leq h_0$, and every $w \in H^2(\Omega^t) \cap \mathcal{V}^t$,

$$\inf_{w_h \in \mathcal{V}_h^t} \|w - w_h\|_{1,2,\Omega^t} \leq Ch|w|_{2,2,\Omega^t}.$$

(3.2.iii) For every $h \leq h_0$, there exists $C > 0$ independent of t such that for every $n = 1, 2, \ldots, N$ and every $t \in (t^{n-1}, t^n)$,

$$\|\dot{v}_h\|_{0,\infty,\Omega^t} \leq C.$$

A statement of the theorem also requires the definition of a bilinear form $\Lambda_h^t : \mathcal{V}^t \times \mathcal{V}^t \to \mathbb{R}$ which embodies the time rate of change of a_h^t, constructed as follows. For a given $t \in (t^{n-1}, t^n]$, let $u, w \in \mathcal{V}^t$. For each $h \leq h_0$, associate with u and w a pair of functions $U \in \mathcal{V}^{t^{n-1}}$ and $W \in \mathcal{V}^{t^{n-1}}$ satisfying

$$U(X) = u(\varphi_h^t(X)), \quad W(X) = w(\varphi_h^t(X))$$

for every $X \in \Omega^{t^{n-1}}$. Now define

$$\Lambda_h^t(u, w) = \left. \frac{d}{d\tau} \right|_{\tau = t} A_h^\tau(U, W), \tag{3.6}$$

where $A_h^\tau : \mathcal{V}^{t^{n-1}} \times \mathcal{V}^{t^{n-1}} \to \mathbb{R}$ denotes the pullback of a_h^τ to $\mathcal{V}^{t^{n-1}} \times \mathcal{V}^{t^{n-1}}$ under the map φ_h^t. That is,

$$A_h^\tau(u \circ \varphi_h^\tau, w \circ \varphi_h^\tau) = a_h^\tau(u, w) \tag{3.7}$$

for every $\tau \in (t^{n-1}, t^n]$ and every $u, w \in \mathcal{V}^\tau$. The (weak) differentiability of A_h^τ in (3.6) is proven in Section 5 under assumptions (2.ii) and (3.2.iii). Note that Λ_h^t is a bilinear form resembling, in loose language, the Lie derivative of a_h^t along the direction of the mesh motion.

To measure the size of the bilinear form Λ_h^t, we make use of the following family of h-dependent norms on the space of continuous bilinear forms $\Lambda : \mathcal{V}^t \times \mathcal{V}^t \to \mathbb{R}$. For $s \geq 0$ and $2 \leq \theta, \eta \leq \infty$ with $\frac{1}{\theta} + \frac{1}{\eta} = \frac{1}{2}$, denote

$$\|\Lambda\|_{-s,\theta,\Omega^t} = \sup_{\substack{u \in W^{1,\eta}(\Omega^t) \cap \mathcal{V}^t \\ w \in H^s(\Omega^t) \cap \mathcal{V}^t \\ u, w \neq 0}} \frac{|\Lambda(u, w)|}{(\|u\|_{0,\eta,\Omega^t} + h|u|_{1,\eta,\Omega^t})\|w\|_{s,2,\Omega^t}}. \tag{3.8}$$

We note in passing that a sample calculation of Λ_h^t and its norms for a particular bilinear form a^t and mesh motion strategy is illustrated in Section 6.

The following theorem will be proved in Section 5.

THEOREM 3.2. *Let $\{\mathcal{V}_h^t \mid 0 < t \leq T,\ h \leq h_0\}$ be a family of evolving finite element spaces with mesh velocity v_h satisfying (3.1.iii) and (3.2.i-3.2.iii). Let ρ, κ, and Λ_h^t be given by (3.4), (3.2), and (3.6), respectively. Then there exists $C > 0$ independent of h and t such that*

$$\begin{aligned}\|D_t\rho\|_{0,2,\Omega^t} \leq C\Big[&h \inf_{w_h \in \mathcal{V}_h^t} \|D_t(e^{-\kappa t}u) - w_h\|_{1,2,\Omega^t} \\ &+ \left(\|\Lambda_h^t\|_{-2,\theta,\Omega^t} + h\|\Lambda_h^t\|_{-1,\theta,\Omega^t}\right)\left(\|\rho\|_{0,\eta,\Omega^t} + h|\rho|_{1,\eta,\Omega^t}\right)\Big]\end{aligned}$$

for every $h \leq h_0$, *a.e.* $t \in (t^{n-1}, t^n)$, *every* $n = 1, 2, \dots, N$, *and every* $2 \leq \theta, \eta \leq \infty$ *satisfying* $\frac{1}{\theta} + \frac{1}{\eta} = \frac{1}{2}$.

The preceding theorem provides an upper bound for $\|D_t \rho\|_{0,2,\Omega^t}$ that can be computed for a given mesh motion and finite element space using knowledge of two properties: the approximation power of the finite element space and the regularity of the mesh's evolution. The regularity of the mesh's evolution is measured in terms of the bilinear form Λ_h^t, which, again, resembles the Lie derivative of a_h^t along the direction of the mesh motion. In determining the order of accuracy of a given scheme, the estimation of the scaling of $\|\Lambda_h^t\|_{-2,\theta,\Omega^t} + h\|\Lambda_h^t\|_{-1,\theta,\Omega^t}$ with respect to h plays a fundamental role, as we illustrate in Section 6.

3.3. Concrete Error Estimate. Theorems 3.1 and 3.2 provide the key ingredients needed to estimate the order of accuracy of numerical methods belonging to the broad class of schemes having the form (2.13). As an illustration, let us consider the case in which the finite element space $\mathcal{V}_h^t$ contains functions which approximate $u(\cdot, t)$ and $D_t u(\cdot, t)$ to order h^r in the L^2-norm, with $r \geq 2$ an integer. To make this precise, and to account for the fact that practical choices of the mesh velocity typically endow $D_t u = \dot{u} + v_h \cdot \nabla_x u$ with lower global regularity than elementwise regularity, let

$$\|u\|_{s,p,\Omega^t,h} = \left(\sum_{K \in \mathcal{T}_h^t} \|u\|_{s,p,K}^p \right)^{1/p}, \quad |u|_{s,p,\Omega^t,h} = \left(\sum_{K \in \mathcal{T}_h^t} |u|_{s,p,K}^p \right)^{1/p}$$

denote the "broken" $W^{s,p}(\Omega^t)$-norm and semi-norm, respectively, for each $s \geq 0$, $1 \leq p \leq \infty$. Define the broken Sobolev spaces

$$W_h^{s,p}(\Omega^t) = \{u \in L^p(\Omega^t) \mid \|u\|_{s,p,\Omega^t,h} < \infty\}$$

and $H_h^s(\Omega^t) = W_h^{s,2}(\Omega^t)$. We shall assume that the finite element spaces $\mathcal{V}_h^t$ satisfy the following approximation hypothesis with an integer $r \geq 2$ and a real number $q \in [1, \infty]$ satisfying $q < \infty$ if $d = 2r - 2$ and $q \leq 2d/(d - 2r + 2)$ if $d > 2r - 2$:

(3.3.i) There exists $C > 0$ independent of h and t such that for every $h \leq h_0$, every $t \in (0, T]$, every $2 \leq s \leq r$, and every $w \in H_h^s(\Omega^t) \cap W^{1,q}(\Omega^t) \cap \mathcal{V}^t$,

$$\inf_{w_h \in \mathcal{V}_h^t} \|w - w_h\|_{m,2,\Omega^t} \leq C h^{s-m} |w|_{s,2,\Omega^t,h}, \quad m = 0, 1. \tag{3.9}$$

Note that this hypothesis can be satisfied, for instance, by a finite element space $\mathcal{V}_h^t$ consisting of continuous functions that are elementwise polynomials of degree $\leq r - 1$ over a shape-regular mesh $\mathcal{T}_h^t$ in dimension $d \leq 3$. In this case, we may take w_h equal to the nodal interpolant of w in (3.9), which is well-defined as long as q is chosen larger than d (so that $W^{1,q}(\Omega^t) \subset C^0(\overline{\Omega^t})$); see [9, Remark 3.2.2].

Finally, let us suppose that the mesh motion strategy is such that the following bounds hold for the number of remeshing times $N(h)$, the bilinear form Λ_h^t, and the mesh velocity v_h:

(3.3.ii) There exists $C > 0$ independent of h such that for every $h \leq h_0$,

$$N(h) \leq C, \tag{3.10}$$

$$\sup_{0 < t \leq T} \|\Lambda_h^t\|_{-2,\infty,\Omega^t} + h\|\Lambda_h^t\|_{-1,\infty,\Omega^t} \leq C, \tag{3.11}$$

$$\sup_{0 < t \leq T} \|v_h\|_{r,\infty,\Omega^t,h} \leq C. \tag{3.12}$$

We remark that hypothesis (3.3.ii) is representative of conventional ALE schemes under mild assumptions on the mesh motion; see Section 6.

The next corollary is then a straightforward consequence of Theorems 3.1 and 3.2 together with classical estimates for $\|\rho(t)\|_{0,2,\Omega^t}$ and $|\rho(t)|_{1,2,\Omega^t}$, which we summarize in Section 5.3.

COROLLARY 3.3. *Let $\{u_h^n\}_{n=0}^N$ be generated by the recursion (2.13) with the initial condition (2.14), using an evolving finite element space $\{\mathcal{V}_h^t \mid 0 < t \le T\}$ satisfying (3.1.iii) and projectors $\{p_h^{t_+^n}\}_{n=0}^{N-1}$ satisfying (3.1.i-3.1.ii). Suppose that (3.2.i-3.2.iii), (3.3.i), and (3.3.ii) hold with an integer $r \ge 2$. Then there exists C independent of h and u such that for every $h \le h_0$, the error $\varepsilon^N = u(\cdot,T) - u_h^N$ satisfies*

$$\|\varepsilon^N\|_{0,2,\Omega^T} \le Ch^r \left(\sup_{0\le t\le T} |u|_{r,2,\Omega^t} + \int_0^T \left(|\dot{u}|_{r,2,\Omega^t} + \|u\|_{r+1,2,\Omega^t} \right) dt \right),$$

provided that for each $t \in [0,T]$, $u(\cdot,t) \in H^{r+1}(\Omega^t)$, and $\dot{u}(\cdot,t) \in H^r(\Omega^t)$.

4. Proof of the Abstract Error Estimate. This section is devoted to the proof of Theorem 3.1.

4.1. Outline of the Proof. After establishing a stability estimate for the semidiscrete flow in Section 4.2 and fixing an appropriate value for the constant κ in Section 4.3, the proof of Theorem 3.1 will proceed in three steps.

First, in Section 4.4, the semidiscrete solution u_h^n is compared with a discrete representative of the exact solution, namely $r_h^{t^n} u^n$. Using standard arguments from the analysis of numerical integrators, the difference $r_h^{t^n} u^n - u_h^n$ is decomposed into a summation of local errors (errors that can be studied over a single interval $(t^{n-1}, t^n]$), each amplified by a power of the projector's stability constant C_p. The decomposition of the error into a summation of local errors is illuminated by Fig. 4.1, where the evolution of the exact and semidiscrete solutions is illustrated schematically. Next, in Section 4.5, the local error at each n is decomposed into two parts that can be understood as an error related to the spatial discretization and an error related to the projection of the semidiscrete solution onto a new finite element space at the start of each interval. These errors can be estimated in terms of the material time derivative of ρ and the jumps in ρ across each remeshing time, respectively. Finally, the aforementioned estimates are combined to yield Theorem 3.1.

4.2. Stability of the Semidiscrete Flow. We start by stating a stability estimate for the semidiscrete advancement operator. In what follows, we denote by

$$\Omega^I = \{(x,t) \mid x \in \Omega^t, t \in I\}$$

the spacetime slab swept out by Ω^t over an interval $I \subset [0,T]$.

LEMMA 4.1. *For every $h \le h_0$, every $1 \le n \le N$, and every $\bar{u}_h \in \mathcal{V}_h^{t_+^{n-1}}$,*

$$\|\mathfrak{f}_h^n \bar{u}_h\|_{0,2,\Omega^{t^n}} \le \|\bar{u}_h\|_{0,2,\Omega^{t^{n-1}}} + \int_{t^{n-1}}^{t^n} \|f\|_{0,2,\Omega^t}\, dt.$$

Proof. Let u_h solve (2.7) with initial condition $u_h(\cdot, t_+^{n-1}) = \bar{u}_h$. Choose $w_h = u_h$ in (2.7) and integrate with respect to time. Noting that $a^t(u_h, u_h) \ge 0$, we obtain

$$\int_{t^{n-1}}^{\tau} \int_{\Omega^t} \frac{\partial}{\partial t} \left(\frac{1}{2} u_h^2 \right) dx\, dt \le \int_{t^{n-1}}^{\tau} \|f\|_{0,2,\Omega^t} \|u_h\|_{0,2,\Omega^t}\, dt$$

for every $\tau \in (t^{n-1}, t^n]$. The regularity $u_h \in \mathcal{V}_h^{(t^{n-1},t^n]}$ implies via Sobolev embeddings that $u_h^2 \in W^{1,p}(\Omega^{(t^{n-1},t^n)})$ with a scalar $p > 1$. Hence, the Gauss-Green theorem [33, Chapter 3, Theorem 6.1] on the spacetime slab $\Omega^{(t^{n-1},\tau)}$ may be applied to give

$$\frac{1}{2}\|u_h\|_{0,2,\Omega^\tau}^2 \leq \frac{1}{2}\|\bar{u}_h\|_{0,2,\Omega^{t^{n-1}}}^2 + \int_{t^{n-1}}^{\tau} \|f\|_{0,2,\Omega^t}\|u_h\|_{0,2,\Omega^t}\,dt$$

for every $\tau \in (t^{n-1}, t^n]$, where we have used the fact that $u_h = 0$ on $\partial\Omega^t$ for every $t \in (t^{n-1}, t^n]$. The result is then an immediate consequence of the following lemma, whose proof is given in, for instance, [7, Lemma A.5]. $\square$

LEMMA 4.2. *Let $y \in C^0([a,b])$ and $g \in L^1(a,b)$ be nonnegative functions on a bounded interval $[a,b] \subset \mathbb{R}$. Suppose that*

$$\frac{1}{2}y(t)^2 \leq \frac{1}{2}y(a)^2 + \int_a^t g(\tau)y(\tau)\,d\tau$$

for every $t \in [a,b]$. Then

$$y(t) \leq y(a) + \int_a^t g(\tau)\,d\tau$$

for every $t \in [a,b]$.

A consequence of Lemma 4.1 and the linearity of (2.7) is that for any $\bar{u}_h, \bar{w}_h \in \mathcal{V}_h^{t_+^{n-1}}$,

$$\|\mathfrak{f}_h^n\bar{u}_h - \mathfrak{f}_h^n\bar{w}_h\|_{0,2,\Omega^{t^n}} \leq \|\bar{u}_h - \bar{w}_h\|_{0,2,\Omega^{t^{n-1}}}. \tag{4.1}$$

Later, we often abuse notation by writing $\mathfrak{f}_h^n\bar{u}_h - \mathfrak{f}_h^n\bar{w}_h = \mathfrak{f}_h^n(\bar{u}_h - \bar{w}_h)$, bearing in mind that the right-hand side tacitly denotes the advancement $(\bar{u}_h - \bar{w}_h)$ with a vanishing source term f. We also make frequent use of the fact that in the absence of a source term f, the operator $\mathfrak{f}_h^n$ is linear.

4.3. Elliptic Projection. As mentioned earlier, our analysis will rely on the use of an elliptic projector associated with a modified bilinear form

$$a_h^t(u,w) = a^t(u,w) - b_h^t(u,w) + \kappa m^t(u,w),$$

with $\kappa \geq 0$ chosen in such a way such that a_h^t is coercive, uniformly in t and h. The following lemma, which is a statement of *Garding's inequality* (see, for example, [6, Theorem 5.6.8]), shows that such a κ exists when (3.1.iii) and (2.2) hold with constants v_{max} and α_0.

LEMMA 4.3. *Let*

$$\kappa = \frac{v_{max}^2}{2\alpha_0}.$$

Then the inequality

$$a_h^t(u,u) \geq \alpha\|u\|_{1,2,\Omega^t}^2 \quad \forall u \in \mathcal{V}^t \tag{4.2}$$

holds for every $t \in (0,T]$ and every $h \leq h_0$ with $\alpha = \alpha_0/2$.

Proof. The proof is a trivial modification of the proof in [6, Theorem 5.6.8], where it is assumed that $a^t(u,u) \geq \alpha_0|u|_{1,2,\Omega^t}^2$ rather than $a^t(u,u) \geq \alpha_0\|u\|_{1,2,\Omega^t}^2$. $\square$

$$\begin{tikzcd}
\cdots & \mathcal{V}^{t^{n-1}} \arrow[r, "i"] \arrow[d, "r_h^{t^{n-1}}"'] & \mathcal{V}^{t_+^{n-1}} \arrow[r, "\mathfrak{f}^n"] \arrow[d, "r_h^{t_+^{n-1}}"'] & \mathcal{V}^{t^n} \arrow[r, "i"] \arrow[d, "r_h^{t^n}"'] & \mathcal{V}^{t_+^n} \arrow[r, "\mathfrak{f}^{n+1}"] \arrow[d, "r_h^{t_+^n}"'] & \mathcal{V}^{t^{n+1}} \arrow[d, "r_h^{t^{n+1}}"'] & \cdots \\
& \mathcal{V}_h^{t^{n-1}} \arrow[r, "p_h^{t_+^{n-1}}"'] & \mathcal{V}_h^{t_+^{n-1}} \arrow[r, "\mathfrak{f}_h^n"'] & \mathcal{V}_h^{t^n} \arrow[r, "p_h^{t_+^n}"'] & \mathcal{V}_h^{t_+^n} \arrow[r, "\mathfrak{f}_h^{n+1}"'] & \mathcal{V}_h^{t^{n+1}} &
\end{tikzcd}$$

FIG. 4.1. *Schematic diagram depicting the evolution of the continuous and semidiscrete solutions u^n and u_h^n, respectively. The semidiscrete solution u_h^n advances to the right along the bottom row of the diagram (after an initial projection onto $\mathcal{V}_h^{0+}$ at $t = 0$) via an alternating sequence of projections $p_h^{t_+^{n-1}} : \mathcal{V}_h^{t^{n-1}} \to \mathcal{V}_h^{t_+^{n-1}}$ and semidiscrete advancements $\mathfrak{f}_h^n : \mathcal{V}_h^{t_+^{n-1}} \to \mathcal{V}_h^{t^n}$. The exact solution advances to the right along the top row of the diagram via an alternating sequence of identity maps $i : \mathcal{V}^{t^{n-1}} \to \mathcal{V}^{t_+^{n-1}}$ and continuous advancements $\mathfrak{f}^n : \mathcal{V}^{t_+^{n-1}} \to \mathcal{V}^{t^n}$, where we have introduced the identity maps and defined the spaces $\mathcal{V}^{t_+^n} := \mathcal{V}^{t^n}$ to facilitate an analogy with the evolution of the semidiscrete solution. If the continuous solution is mapped onto the current finite element space via the elliptic projection $r_h^t : \mathcal{V}^t \to \mathcal{V}_h^t$ (vertical arrows), then the difference $r_h^{t^n} u^n - u_h^n$ measures the extent to which this diagram fails to commute.*

It is clear that with κ so defined, a_h^t is continuous, uniformly in t and h. That is, there exists $M > 0$ independent of h and t such that for every $t \in (0, T]$, every $h \le h_0$, and every $u, w \in \mathcal{V}^t$,

$$|a_h^t(u, w)| \le M \|u\|_{1,2,\Omega^t} \|w\|_{1,2,\Omega^t}.$$

Define for each $t \in (0, T]$ the *elliptic projector* $r_h^t : \mathcal{V}^t \to \mathcal{V}_h^t$ according to (3.3). It is a consequence of the Lax-Milgram theorem that r_h^t is a well-defined linear projector for each t.

4.4. Error Decomposition. To derive an estimate for the global error $\varepsilon^n = u^n - u_h^n$, let us compare u_h^n with a discrete representative of the exact solution, namely $r_h^{t^n} u^n$, by writing

$$\varepsilon^n = -e^{\kappa t^n} \rho(t^n) + \ell^n,$$

with ρ given by (3.4) and

$$\ell^n = r_h^{t^n} u^n - u_h^n.$$

Next, decompose ℓ^n as $\ell^n = \ell_1^n + \ell_2^n$ with

$$\begin{aligned}
\ell_1^n &= r_h^{t^n} u^n - \mathfrak{f}_h^n p_h^{t_+^{n-1}} r_h^{t^{n-1}} u^{n-1}, \\
\ell_2^n &= \mathfrak{f}_h^n p_h^{t_+^{n-1}} r_h^{t^{n-1}} u^{n-1} - u_h^n.
\end{aligned}$$

Note that r_h^0 is undefined; in the relations above, it is to be understood that $r_h^0 u^0 = u^0$, so that when $n = 1$,

$$\begin{aligned}
\ell_1^1 &= r_h^{t^1} u^1 - \mathfrak{f}_h^1 p_h^{0+} u^0 = r_h^{t^1} u^1 - u_h^1, \\
\ell_2^1 &= \mathfrak{f}_h^1 p_h^{0+} u^0 - u_h^1 = 0.
\end{aligned}$$

The linearity of $\mathfrak{f}_h^n$ (in the absence of a source term) and the linearity of $p_h^{t_+^{n-1}}$ imply that

$$\begin{aligned}\ell_2^n &= \mathfrak{f}_h^n p_h^{t_+^{n-1}} \left(r_h^{t^{n-1}} u^{n-1} - u_h^{n-1} \right) \\ &= \mathfrak{f}_h^n p_h^{t_+^{n-1}} \left(\ell_1^{n-1} + \ell_2^{n-1} \right).\end{aligned}$$

The stability assumption (3.1.ii) and the stability estimate (4.1) then imply

$$\|\ell_2^n\|_{0,2,\Omega^{t^n}} \le C_p \left(\|\ell_1^{n-1}\|_{0,2,\Omega^{t^{n-1}}} + \|\ell_2^{n-1}\|_{0,2,\Omega^{t^{n-1}}} \right).$$

Combining this recursion with the initial condition $\ell_2^1 = 0$ leads to the bound

$$\|\ell^n\|_{0,2,\Omega^T} \le \sum_{n=1}^{N} C_p^{N-n} \|\ell_1^n\|_{0,2,\Omega^{t^n}}. \tag{4.3}$$

4.5. Estimates for Local Errors. To estimate the local errors ℓ_1^n, $n = 1, 2, \ldots, N$, write

$$\ell_1^n = \gamma^n + \delta^n$$

with

$$\begin{aligned}\gamma^n &= r_h^{t^n} u^n - \mathfrak{f}_h^n r_h^{t_+^{n-1}} u^{n-1}, \\ \delta^n &= \mathfrak{f}_h^n r_h^{t_+^{n-1}} u^{n-1} - \mathfrak{f}_h^n p_h^{t_+^{n-1}} r_h^{t^{n-1}} u^{n-1}.\end{aligned}$$

Again, it is to be understood that $r_h^0 u^0 = u^0$, so that

$$\delta^1 = \mathfrak{f}_h^1 \left(r_h^{0_+} u^0 - p_h^{0_+} u^0 \right).$$

To bound δ^n, use the linearity of $\mathfrak{f}_h^n$ (in the absence of a source term) together with the linearity and surjectivity of $p_h^{t_+^{n-1}}$ to write

$$\begin{aligned}\delta^n &= \mathfrak{f}_h^n p_h^{t_+^{n-1}} \left(r_h^{t_+^{n-1}} - r_h^{t^{n-1}} \right) u^{n-1} \\ &= e^{\kappa t^{n-1}} \mathfrak{f}_h^n p_h^{t_+^{n-1}} \left(\rho(t_+^{n-1}) - \rho(t^{n-1}) \right).\end{aligned}$$

Now by the stability assumption (3.1.ii) and the stability estimate (4.1),

$$\|\delta^n\|_{0,2,\Omega^{t^n}} \le C_p e^{\kappa t^{n-1}} \|\rho(t_+^{n-1}) - \rho(t^{n-1})\|_{0,2,\Omega^{t^{n-1}}}. \tag{4.4}$$

Finally, a bound on γ^n is given in the following lemma.

LEMMA 4.4. *For every $1 \le n \le N$,*

$$\|\gamma^n\|_{0,2,\Omega^{t^n}} \le \int_{t^{n-1}}^{t^n} e^{\kappa t} \|D_t \rho\|_{0,2,\Omega^t} \, dt. \tag{4.5}$$

Proof. Let y_h denote the solution to (2.8) over $(t^{n-1}, t^n]$ with the initial condition

$$y_h(\cdot, t_+^{n-1}) = r_h^{t_+^{n-1}} u^{n-1},$$

so that

$$y_h(\cdot, t^n) = \mathfrak{f}_h^n r_h^{t_+^{n-1}} u^{n-1}.$$

Then, denoting $\theta_h(t) = y_h(\cdot, t) - r_h^t u(\cdot, t)$, we have

$$\begin{aligned} \theta_h(t_+^{n-1}) &= 0, \\ \theta_h(t^n) &= -\gamma^n, \end{aligned}$$

and

$$y_h(\cdot, t) - u(\cdot, t) = \theta_h(t) + e^{\kappa t}\rho(t).$$

A bound on $\theta_h(t^n)$, and hence γ^n, follows from subtracting (2.10) from (2.8) with $w = w_h \in \mathcal{V}_h^t$ and simplifying, using (3.1) together with the equalities

$$a_h^t(\rho, w_h) = 0$$

and

$$D_t(e^{\kappa t}\rho) = \kappa e^{\kappa t}\rho + e^{\kappa t} D_t\rho.$$

The resulting differential equation for θ_h reads

$$m^t(\dot{\theta}_h, w_h) + a^t(\theta_h, w_h) = -m^t(e^{\kappa t} D_t\rho, w_h)$$

for every $w_h \in \mathcal{V}_h^t$, $t \in (t^{n-1}, t^n]$. Lemma 4.1 then gives

$$\|\gamma^n\|_{0,2,\Omega^{t^n}} = \|\mathfrak{f}_h^n \theta_h(t_+^{n-1})\|_{0,2,\Omega^{t^n}} \leq \int_{t^{n-1}}^{t^n} e^{\kappa t} \|D_t\rho\|_{0,2,\Omega^t}\, dt.$$

□

Combining the bounds (4.3), (4.4), and (4.5) leads to the general error estimate in Theorem 3.1.

5. Proof of the Bound on $D_t\rho$. This section presents a proof of Theorem 3.2, which concerns the material time derivative of $\rho = e^{-\kappa t}(r_h^t u - u)$, the (scaled) difference between u and its elliptic projection $r_h^t u$ onto an evolving finite element space $\{\mathcal{V}_h^t\}_t$ with respect to the bilinear form (3.1).

To prove Theorem 3.2, we derive in Lemma 5.4 an equation that relates the material time derivative of ρ to the function ρ itself. Deriving this relation involves some preliminary calculations that lead to a formula in Lemma 5.2 for the time derivative of a time-dependent bilinear form. The relation between $D_t\rho$ and ρ that we derive in Lemma 5.4 then leads to an estimate for the H^1-norm of $D_t\rho$ in Lemma 5.5. Finally, we use a duality argument to estimate the L^2-norm of $D_t\rho$, thereby proving Theorem 3.2. Corollary 3.3 will then follow readily using classical estimates for $\|\rho\|_{0,2,\Omega^t}$ and $|\rho|_{1,2,\Omega^t}$, as we explain in Section 5.3.

5.1. Differentiating the Bilinear Form. We begin by presenting a statement of the Reynolds transport theorem for weakly differentiable functions.

LEMMA 5.1. *Let $\{\varphi^t : \Omega^{\tau_0} \to \Omega^t \mid \tau_0 < t \leq \tau_1\}$ be a regular domain deformation with velocity $v(\cdot, t) = \dot{\varphi}^t \circ (\varphi^t)^{-1}$ over an interval $(\tau_0, \tau_1] \subset [0, T]$. Let $g : \Omega^{(\tau_0,\tau_1]} \to \mathbb{R}$ be such that the map $t \mapsto g(\varphi^t(\cdot), t)$ belongs to $W^{1,p}(\tau_0, \tau_1; L^1(\Omega^{\tau_0}))$ for some $1 \leq p \leq \infty$. Then the map*

$$t \mapsto \int_{\Omega^t} g \, dx$$

belongs to $W^{1,p}(\tau_0, \tau_1)$ and satisfies

$$\frac{d}{dt} \int_{\Omega^t} g \, dx = \int_{\Omega^t} D_t g + g \nabla_x \cdot v \, dx$$

for a.e. $t \in (\tau_0, \tau_1)$.

Proof. A proof of this identity when $g \in W^{1,1}(\Omega^{(\tau_0,\tau_1)})$ is outlined in [5, Lemma 2.2]. A similar proof applies to the case in which $t \mapsto g(\varphi^t(\cdot), t) \in W^{1,p}(\tau_0, \tau_1; L^1(\Omega^{\tau_0}))$. □

Let us now fix a family of evolving finite element spaces $\{\mathcal{V}_h^t \mid 0 < t \leq T\}$, $h \leq h_0$, over a family of moving meshes $\{\mathcal{T}_h^t \mid 0 < t \leq T\}$, $h \leq h_0$, associated with regular domain deformations $\varphi_h^t : \Omega^{t^{n-1}} \to \Omega^t$, $t \in (t^{n-1}, t^n]$, $n = 1, 2, \dots, N$. We assume throughout the remainder of this section that (3.2.iii) holds. Our aim at the moment is to derive a formula for the time-derivative of $a_h^t(u, w)$ for a pair of functions $u, w : \Omega^{(t^{n-1},t^n]} \to \mathbb{R}$ whose regularity will be specified shortly.

The hypothesis (2.ii) ensures that if $u = U \circ (\varphi_h^t)^{-1}$ and $w = W \circ (\varphi_h^t)^{-1}$ for some functions $U, W \in \mathcal{V}^{t^{n-1}}$, then the map $t \mapsto a^t(u, w)$ is Lipschitz, and hence weakly differentiable. Lemma 5.1 reveals that for such functions u and w, the maps $t \mapsto b_h^t(u, w)$ and $t \mapsto m^t(u, w)$ (and hence $t \mapsto a_h^t(u, w)$) are likewise Lipschitz, satisfying a bound of the form (2.4). This can be checked by choosing $g = (v_h \cdot \nabla_x u) w$ and $g = uw$, respectively, in Lemma 5.1, bearing in mind that (3.2.iii) is assumed to hold. It follows, in particular, that the bilinear form Λ_h^t given by (3.6) exists and is continuous for a.e. t, with a modulus of continuity bounded uniformly in time. The next lemma examines the time-differentiability of $a_h^t(u, w)$ for more general u and w.

LEMMA 5.2. *Let $u, w : \Omega^{(t^{n-1},t^n]} \to \mathbb{R}$ be such that the maps $t \mapsto u(\varphi_h^t(\cdot), t)$ and $t \mapsto w(\varphi_h^t(\cdot), t)$, $t \in (t^{n-1}, t^n]$, belong to $W^{1,1}(t^{n-1}, t^n; \mathcal{V}^{t^{n-1}})$. Then for every $h \leq h_0$, the map*

$$t \mapsto a_h^t(u, w)$$

belongs to $W^{1,1}(t^{n-1}, t^n)$ and satisfies

$$\frac{d}{dt} a_h^t(u, w) = a_h^t(D_t u, w) + a_h^t(u, D_t w) + \Lambda_h^t(u, w) \tag{5.1}$$

for a.e. $t \in (t^{n-1}, t^n)$, where Λ_h^t is given by (3.6).

Proof. Let $U(t) := u(\varphi_h^t(\cdot), t)$ and $W(t) := w(\varphi_h^t(\cdot), t)$. Note that $U, W \in C([t^{n-1}, t^n], \mathcal{V}^{t^{n-1}})$ by the embedding $W^{1,1}(t^{n-1}, t^n; \mathcal{V}^{t^{n-1}}) \subset C([t^{n-1}, t^n], \mathcal{V}^{t^{n-1}})$ [14, Chapter 5.9, Theorem 2]. Using mollification in time, there exist sequences of infinitely differentiable $\mathcal{V}^{t^{n-1}}$-valued functions $U_\epsilon, W_\epsilon : (-\infty, \infty) \to \mathcal{V}^{t^{n-1}}$ with compact support whose restrictions to $[t^{n-1}, t^n]$ converge to U and W uniformly on $[t^{n-1}, t^n]$ as

$\epsilon \to 0$, and whose derivatives converge to $\frac{\partial U}{\partial t}$ and $\frac{\partial W}{\partial t}$ in $L^1(t^{n-1}, t^n; \mathcal{V}^{t^{n-1}})$ as $\epsilon \to 0$. Define

$$p_\epsilon(\sigma, \tau) = A_h^\tau(U_\epsilon(\sigma), W_\epsilon(\sigma)), \quad \sigma, \tau \in (t^{n-1}, t^n),$$

where $A_h^t : \mathcal{V}^{t^{n-1}} \times \mathcal{V}^{t^{n-1}} \to \mathbb{R}$ denotes the pullback of a_h^t under φ_h^t, as defined in (3.7). The remarks preceding this lemma imply that for a.e. $\tau \in (t^{n-1}, t^n)$, $\frac{\partial p_\epsilon}{\partial \tau}(\sigma, \tau)$ exists and is given by

$$\frac{\partial p_\epsilon}{\partial \tau}(\sigma, \tau) = \Lambda_h^\tau \left(U_\epsilon(\sigma) \circ (\varphi_h^\tau)^{-1}, W_\epsilon(\sigma) \circ (\varphi_h^\tau)^{-1} \right)$$

for every $\sigma \in (t^{n-1}, t^n)$. Furthermore, since A_h^τ is a continuous bilinear form for each τ, the equality

$$\frac{\partial p_\epsilon}{\partial \sigma}(\sigma, \tau) = A_h^\tau \left(\frac{\partial U_\epsilon}{\partial \sigma}(\sigma), W_\epsilon(\sigma) \right) + A_h^\tau \left(U_\epsilon(\sigma), \frac{\partial W_\epsilon}{\partial \sigma}(\sigma) \right)$$

holds for every $(\sigma, \tau) \in (t^{n-1}, t^n) \times (t^{n-1}, t^n)$ [29, Theorem 2.4.4]. Notice that $\frac{\partial p_\epsilon}{\partial \sigma}$ is continuous in $(t^{n-1}, t^n) \times (t^{n-1}, t^n)$ by virtue of the (temporally uniform) continuity of A_h^t and Λ_h^t, and the regularity of φ_h^t, U_ϵ, and W_ϵ. It follows [2, Theorem 12.11] that $p_\epsilon(\sigma, \tau)$ is differentiable at $(\sigma, \tau) = (t, t)$ for a.e. $t \in (t^{n-1}, t^n)$ and satisfies

$$\frac{d}{dt} p_\epsilon(t, t) = \left. \frac{\partial}{\partial \sigma} \right|_{\sigma = t} p_\epsilon(\sigma, t) + \left. \frac{\partial}{\partial \tau} \right|_{\tau = t} p_\epsilon(t, \tau).$$

Now multiply by a smooth real-valued function with compact support in (t^{n-1}, t^n) and integrate in time. Integrating by parts and taking the limit as $\epsilon \to 0$ shows that the equality

$$\begin{aligned} \frac{d}{dt} A_h^t(U(\cdot, t), W(\cdot, t)) = A_h^t \left(\frac{\partial U}{\partial t}(\cdot, t), W(\cdot, t) \right) + A_h^t \left(U(\cdot, t), \frac{\partial W}{\partial t}(\cdot, t) \right) \\ + \Lambda_h^t \left(u(\cdot, t), w(\cdot, t) \right) \end{aligned}$$

holds in the sense of distributions. Conclude using the definition of A_h^t together with the relations

$$\begin{aligned} \frac{\partial U}{\partial t}(\cdot, t) &= D_t u(\varphi_h^t(\cdot), t), \\ \frac{\partial W}{\partial t}(\cdot, t) &= D_t w(\varphi_h^t(\cdot), t). \end{aligned}$$

□

5.2. Estimating $D_t\rho$**.** We now use Lemma 5.2 to derive a relation between $\rho = e^{-\kappa t}(r_h^t u - u)$ and its material time derivative. In order to justify the forthcoming calculations, we first make the following observation concerning the regularity of ρ. A proof is given in Appendix A.

LEMMA 5.3. *For each $n = 1, 2, \dots, N$ the map $t \mapsto \rho(t) \circ \varphi_h^t$, $t \in (t^{n-1}, t^n]$, belongs to $W^{1,1}(t^{n-1}, t^n, \mathcal{V}^{t^{n-1}})$.*

One consequence of the preceding lemma is that $D_t\rho \in \mathcal{V}^t$ for a.e. $t \in (t^{n-1}, t^n)$. We tacitly make use of the regularity of ρ in the following lemma.

LEMMA 5.4. *For every $h \leq h_0$, every $n = 1, 2, \dots, N$, and a.e. $t \in (t^{n-1}, t^n)$, it holds that*

$$a_h^t(D_t\rho, w_h) = -\Lambda_h^t(\rho, w_h) \quad \forall w_h \in \mathcal{V}_h^t. \tag{5.2}$$

Proof. For a given $w_h \in \mathcal{V}_h^t$, take $w(\cdot,\tau) = w_h \circ \varphi_h^t \circ (\varphi_h^\tau)^{-1}$ and $u(\cdot,\tau) = \rho(\tau)$ for each $\tau \in (t^{n-1}, t^n]$ in (5.1). Then use the relation

$$a_h^t(\rho, w) = 0 \quad \forall t$$

together with the fact that the material time derivative of w is zero. ☐

The relation (5.2) leads to the following estimate for the H^1-norm of the material time derivative of ρ.

LEMMA 5.5. *There exists $C > 0$ independent of h and t such that*

$$\|D_t\rho\|_{1,2,\Omega^t} \le C\left(\inf_{w_h\in\mathcal{V}_h^t} \|D_t(e^{-\kappa t}u) - w_h\|_{1,2,\Omega^t} + \|\Lambda_h^t\|_{-1,\theta,\Omega^t}\left(\|\rho\|_{0,\eta,\Omega^t} + h|\rho|_{1,\eta,\Omega^t}\right)\right)$$

for every $h \le h_0$, a.e. $t \in (t^{n-1}, t^n)$, every $n = 1, 2, \dots, N$, and every $2 \le \theta, \eta \le \infty$ such that $\frac{1}{\theta} + \frac{1}{\eta} = \frac{1}{2}$.

Proof. The coercivity of a_h^t and the relation (5.2) imply that

$$\begin{aligned}\alpha\|D_t\rho\|_{1,2,\Omega^t}^2 &\le a_h^t(D_t\rho, D_t\rho)\\ &= a_h^t(D_t\rho, D_t\rho - w_h) + \Lambda_h^t(\rho, D_t\rho - w_h) - \Lambda_h^t(\rho, D_t\rho)\end{aligned}$$

for any $w_h \in \mathcal{V}_h^t$. Now by the continuity of a_h^t and the definition (3.8),

$$\begin{aligned}\alpha\|D_t\rho\|_{1,2,\Omega^t}^2 \le\; &M\|D_t\rho\|_{1,2,\Omega^t}\|D_t\rho - w_h\|_{1,2,\Omega^t}\\ &+ \|\Lambda_h^t\|_{-1,\theta,\Omega^t}\left(\|\rho\|_{0,\eta,\Omega^t} + h|\rho|_{1,\eta,\Omega^t}\right)\left(\|D_t\rho - w_h\|_{1,2,\Omega^t} + \|D_t\rho\|_{1,2,\Omega^t}\right)\end{aligned}$$

for any $2 \le \theta, \eta \le \infty$ with $\frac{1}{\theta} + \frac{1}{\eta} = \frac{1}{2}$. Using the fact that for real numbers $x, a, b \ge 0$,

$$x^2 \le ax + bx + ab \implies x \le \frac{1+\sqrt{2}}{2}(a+b),$$

it follows that

$$\|D_t\rho\|_{1,2,\Omega^t} \le C\left(\inf_{w_h\in\mathcal{V}_h^t} \|D_t\rho - w_h\|_{1,2,\Omega^t} + \|\Lambda_h^t\|_{-1,\theta,\Omega^t}\left(\|\rho\|_{0,\eta,\Omega^t} + h|\rho|_{1,\eta,\Omega^t}\right)\right),$$

with C depending only on M and α. Finally, observe that $D_t(e^{-\kappa t}r_h^t u) = D_t(r_h^t(e^{-\kappa t}u)) \in \mathcal{V}_h^t$. ☐

We shall now use a duality argument to derive an estimate for the L^2-norm of the material time derivative of ρ, thereby proving Theorem 3.2. To this end, suppose that (3.2.i-3.2.ii) hold and let $y \in \mathcal{V}^t$ solve the adjoint problem

$$a_h^t(w, y) = m^t(w, D_t\rho) \quad \forall w \in \mathcal{V}^t.$$

Observe that

$$\begin{aligned}m^t(D_t\rho, D_t\rho) &= a_h^t(D_t\rho, y)\\ &= a_h^t(D_t\rho, y - w_h) + \Lambda_h^t(\rho, y - w_h) - \Lambda_h^t(\rho, y)\end{aligned}$$

for any $w_h \in \mathcal{V}_h^t$. Hence,

$$\begin{aligned}\|D_t\rho\|_{0,2,\Omega^t}^2 \le\; &M\|D_t\rho\|_{1,2,\Omega^t} \inf_{w_h\in\mathcal{V}_h^t} \|y - w_h\|_{1,2,\Omega^t}\\ &+ \|\Lambda_h^t\|_{-1,\theta,\Omega^t}\left(\|\rho\|_{0,\eta,\Omega^t} + h|\rho|_{1,\eta,\Omega^t}\right) \inf_{w_h\in\mathcal{V}_h^t} \|y - w_h\|_{1,2,\Omega^t}\\ &+ \|\Lambda_h^t\|_{-2,\theta,\Omega^t}\left(\|\rho\|_{0,\eta,\Omega^t} + h|\rho|_{1,\eta,\Omega^t}\right)\|y\|_{2,2,\Omega^t}\end{aligned}$$

The theorem then follows from Lemma 5.5, hypothesis (3.2.ii), and the elliptic regularity estimate (3.5).

5.3. Deducing the Concrete Error Estimate. Deducing Corollary 3.3 is now a matter of using (3.3.i) and (3.3.ii) to simplify the bound in Theorem 3.2, inserting the resulting bound for $\|D_t\rho\|_{0,2,\Omega^t}$ into the inequality in Theorem 3.1, and invoking estimates for $\|\rho\|_{0,2,\Omega^t}$, $|\rho|_{1,2,\Omega^t}$, and $\|\rho(t^n_+)-\rho(t^n)\|_{0,2,\Omega^{t^n}}$ which we summarize below. We assume in what follows that u and v_h satisfy the regularity assumptions made in the statement of Corollary 3.3.

Estimates for ρ. By the (temporally uniform) coercivity and continuity of a^t_h, it follows from classical arguments (namely, via Céa's Lemma [13, Lemma 2.28], the Aubin-Nitsche Lemma [13, Lemma 2.31], (3.3.i), and (3.2.i)) that there exists $C > 0$ independent of h and t such that

$$e^{\kappa t}\|\rho(t)\|_{m,2,\Omega^t} \leq Ch^{s-m}|u|_{s,2,\Omega^t}, \quad m = 0, 1, \tag{5.3}$$

for every $2 \leq s \leq r$, every $0 < t \leq T$, and every $h \leq h_0$.

Estimate for $\rho(t^n_+)-\rho(t^n)$. The triangle inequality and (5.3) (with $m = 0$) provide the following upper bound for the jumps in ρ across the times of remeshing: For every $2 \leq s \leq r$, every $n = 1, 2, \dots, N$, and every $h \leq h_0$,

$$e^{\kappa t^n}\|\rho(t^n_+) - \rho(t^n)\|_{0,2,\Omega^{t^n}} \leq Ch^s|u|_{s,2,\Omega^{t^n}}. \tag{5.4}$$

We remark that sharper estimates for $\rho(t^n_+) - \rho(t^n)$ may hold when the meshes $\mathcal{T}_h^{t^n_+}$ and $\mathcal{T}_h^{t^n}$ coincide over a large fraction of the domain, though we do not address this situation here. This phenomenon is the subject of [22] and plays a role in error estimates for universal meshes.

To simplify the bound in Theorem 3.2 and thereby obtain Corollary 3.3, expand

$$\begin{aligned} D_t(e^{-\kappa t}u) &= \partial_t(e^{-\kappa t}u) + v_h \cdot \nabla_x(e^{-\kappa t}u) \\ &= e^{-\kappa t}\dot{u} - \kappa e^{-\kappa t}u + e^{-\kappa t}v_h \cdot \nabla_x u. \end{aligned}$$

Now use the facts that $\dot{u}(\cdot,t) \in H^r(\Omega^t)$, $u(\cdot,t) \in H^{r+1}(\Omega^t)$, $\nabla_x u(\cdot,t) \in H^r(\Omega^t)^d \subset W^{1,q}(\Omega^t)^d$, and $v_h(\cdot,t) \in W^{1,\infty}(\Omega^t) \cap W^{r,\infty}_h(\Omega^t)$ to deduce that $D_t(e^{-\kappa t}u) \in H^r_h(\Omega^t) \cap W^{1,q}(\Omega^t) \cap \mathcal{V}^t$ for any $q \in [1,\infty]$ satisfying $q < \infty$ if $d = 2r-2$ and $q \leq 2d/(d-2r+2)$ if $d > 2r-2$. Thus, by (3.3.i),

$$\begin{aligned} e^{\kappa t} \inf_{w_h \in \mathcal{V}^t_h} \|D_t(e^{-\kappa t}u) - w_h\|_{1,2,\Omega^t} &\leq Ch^{r-1}|\dot{u} - \kappa u + v_h \cdot \nabla_x u|_{r,2,\Omega^t,h} \\ &\leq Ch^{r-1}\left(|\dot{u}|_{r,2,\Omega^t,h} + \kappa|u|_{r,2,\Omega^t,h} + |v_h \cdot \nabla_x u|_{r,2,\Omega^t,h}\right) \\ &= Ch^{r-1}\left(|\dot{u}|_{r,2,\Omega^t} + \kappa|u|_{r,2,\Omega^t} + |v_h \cdot \nabla_x u|_{r,2,\Omega^t,h}\right). \end{aligned} \tag{5.5}$$

Next, use the fact that, with a constant C depending only on r and d, it holds that

$$\begin{aligned} |v_h \cdot \nabla_x u|_{r,2,\Omega^t,h} &\leq C\|v_h\|_{r,\infty,\Omega^t,h}\|u\|_{r+1,2,\Omega^t,h} \\ &= C\|v_h\|_{r,\infty,\Omega^t,h}\|u\|_{r+1,2,\Omega^t}. \end{aligned} \tag{5.6}$$

The proof of Corollary 3.3 is completed by combining Thoerems 3.1 and 3.2 with (5.3), (5.4), (5.5), and (5.6).

6. Applications to Specific Mesh Motion Strategies.

6.1. Application to ALE Schemes. In this section, we verify the hypotheses of Corollary 3.3 for a more concrete mesh motion strategy and bilinear form a^t. The situation we have in mind is that in which the mesh motion is associated with an arbitrary Lagrangian-Eulerian (ALE) scheme. Such a scheme typically prescribes a motion of the mesh by choosing a mesh for Ω^0 and solving a global system of equations (such as those of linear elasticity) for the nodal positions at times $t > 0$, remeshing as often as needed to maintain a mesh of adequate quality. Rather than considering an explicit instance of such a method, we leave the precise choice of the mesh deformation unspecified and simply provide an example of an assumption on the mesh deformation that ensures optimal order of convergence.

The assumption we make is that the mesh velocity v_h approximates a smooth velocity v in the following sense.

(6.i) There exists $v : \Omega \to \mathbb{R}^d$ and constants $C_i(v)$, $i = 1, 2, 3$, independent of h and t such that for every $t \in (t^{n-1}, t^n)$ and every $n = 1, 2, \ldots, N$,

$$\|v_h - v\|_{1,\infty,\Omega^t} \le C_1(v)h, \quad \|\dot{v}_h - \dot{v}\|_{0,\infty,\Omega^t} \le C_2(v)h,$$

and

$$C_3(v) := \max_{1\le n\le N} \sup_{t\in(t^{n-1},t^n)} \|v\|_{2,\infty,\Omega^t} + \|\dot{v}\|_{1,\infty,\Omega^t} < \infty.$$

We consider the case in which the bilinear form a^t is given by

$$a^t(u, w) = \int_{\Omega^t} \nabla_x u \cdot \nabla_x w \, dx. \tag{6.1}$$

By virtue of the Poincaré inequality and the boundedness of the spacetime domain Ω, this bilinear form is coercive and continuous, uniformly in time. Furthermore, its satisfaction of condition (2.ii) can be inferred from Lemma 6.2 below.

Our aim in this section is to show that the bilinear form Λ_h^t (defined in (3.6)) in this setting satisfies a bound of the form (3.11), thereby validating the conditions of Corollary 3.3 for such a mesh motion strategy whenever (3.10) and (3.12) hold as well. For numerical illustrations of the optimal convergence rates predicted by Corollary 3.3, we refer the reader to examples in the existing literature on ALE schemes, such as [31, 30].

We begin by presenting an explicit formula for Λ_h^t when a^t is given by (6.1). A proof of this result is given in Appendix A.

LEMMA 6.1. *Let a^t and Λ_h^t be given by (6.1) and (3.6), respectively. Then Λ_h^t satisfies*

$$\begin{aligned}\Lambda_h^t(u, w) = &-\int_{\Omega^t} \nabla_x u \cdot \left(\nabla_x v_h + (\nabla_x v_h)^\dagger\right) \nabla_x w \, dx - \int_{\Omega^t} (\dot{v}_h \cdot \nabla_x u) w \, dx \\ &+ \int_{\Omega^t} (\nabla_x u \cdot \nabla_x w) \nabla_x \cdot v_h \, dx - \int_{\Omega^t} (v_h \cdot \nabla_x u) w \nabla_x \cdot v_h \, dx + \kappa \int_{\Omega^t} u w \nabla_x \cdot v_h \, dx\end{aligned} \tag{6.2}$$

for every $h \le h_0$, a.e. $t \in (t^{n-1}, t^n)$, every $n = 1, 2, \ldots, N$, and every $u, w \in \mathcal{V}^t$, where $(\nabla_x v_h)^\dagger$ denotes the adjoint of $\nabla_x v_h$.

We conclude with estimates for $\|\Lambda_h^t\|_{-1,\infty,\Omega^t}$ and $\|\Lambda_h^t\|_{-2,\infty,\Omega^t}$, which imply (3.11) when combined.

LEMMA 6.2. *Suppose that (6.i) holds. Then there exists C independent of h and t such that the bilinear form (6.2) satisfies*

$$\|\Lambda_h^t\|_{-1,\infty,\Omega^t} \le Ch^{-1}$$

for every $h \leq h_0$, a.e. $t \in (t^{n-1}, t^n)$, and every $n = 1, 2, \ldots, N$.

Proof. Let $u, w \in \mathcal{V}^t$. Apply the Cauchy-Schwarz inequality to each term in (6.2) to obtain

$$|\Lambda_h^t(u,w)| \leq C\left((1+\kappa+v_{max})|v_h|_{1,\infty,\Omega^t} + \|\dot{v}_h\|_{0,\infty,\Omega^t}\right)\|u\|_{1,2,\Omega^t}\|w\|_{1,2,\Omega^t}.$$

Then use hypothesis (6.i) to bound $|v_h|_{1,\infty,\Omega^t}$ and $\|\dot{v}_h\|_{0,\infty,\Omega^t}$ uniformly in h and t. Finally, use the fact that

$$\|u\|_{1,2,\Omega^t} \leq h^{-1}\max\{1, h_0\}(\|u\|_{0,2,\Omega^t} + h|u|_{1,2,\Omega^t})$$

to deduce that

$$|\Lambda_h^t(u,w)| \leq Ch^{-1}(\|u\|_{0,2,\Omega^t} + h|u|_{1,2,\Omega^t})\|w\|_{1,2,\Omega^t}$$

with a constant C independent of h and t. ◻

LEMMA 6.3. *Suppose that (6.i) holds. Then there exists C independent of h and t such that bilinear form (6.2) satisfies*

$$\|\Lambda_h^t\|_{-2,\infty,\Omega^t} \leq C$$

for every $h \leq h_0$, a.e. $t \in (t^{n-1}, t^n)$, and every $n = 1, 2, \ldots, N$.

Proof. Let $u \in \mathcal{V}^t$ and $w \in H^2(\Omega^t) \cap \mathcal{V}^t$. Define

$$\begin{aligned}\Lambda^t(u,w) = &-\int_{\Omega^t} \nabla_x u \cdot \left(\nabla_x v + (\nabla_x v)^\dagger\right)\nabla_x w\,dx - \int_{\Omega^t}(\dot{v}\cdot\nabla_x u)w\,dx \\ &+ \int_{\Omega^t}(\nabla_x u\cdot\nabla_x w)\,\nabla_x\cdot v\,dx - \int_{\Omega^t}(v\cdot\nabla_x u)w\nabla_x\cdot v\,dx + \kappa\int_{\Omega^t} uw\nabla_x\cdot v\,dx,\end{aligned} \tag{6.3}$$

where v denotes the smooth vector field described in (6.i). A straightforward calculation gives

$$|\Lambda_h^t(u,w) - \Lambda^t(u,w)| \leq Ch\|u\|_{1,2,\Omega^t}\|w\|_{1,2,\Omega^t}$$

with C depending only on κ, v_{max}, and the constants $C_i(v)$, $i = 1, 2, 3$, appearing in (6.i). On the other hand, integrating each term of (6.3) except the last by parts leads to the bound

$$|\Lambda^t(u,w)| \leq C\|u\|_{0,2,\Omega^t}\|w\|_{2,2,\Omega^t}.$$

with C depending only on κ and $C_3(v)$. The conclusion then follows from

$$\begin{aligned}|\Lambda_h^t(u,w)| &\leq |\Lambda^t(u,w)| + |\Lambda_h^t(u,w) - \Lambda^t(u,w)| \\ &\leq C\left(\|u\|_{0,2,\Omega^t} + h|u|_{1,2,\Omega^t}\right)\|w\|_{2,2,\Omega^t}.\end{aligned}$$

◻

6.2. Application to Universal Meshes. We now briefly discuss another situation to which Theorems 3.1 and 3.2 may be applied, namely, to a mesh motion obtained from a *universal mesh* [21, 20, 32]. This strategy utilizes a background triangulation of an ambient domain $\mathcal{D} \supset \Omega^t$, $0 \leq t \leq T$, to construct a conforming mesh for the immersed domain at all times using small deformations of a periodically updated reference subtriangulation of the background mesh. In that setting, the number N of "remeshing" times (the periodic updates of the reference subtriangulation) scales like h^{-1}, the mesh velocity is nonzero only in a region of measure $O(h)$, and its spatial gradient scales like h^{-1}. When such a strategy is used with piecewise polynomial finite elements of degree $\leq r-1$, we expect based on preliminary calculations that (under suitable regularity assumptions) the quantity $\|D_t\rho\|_{0,2,\Omega^t}$ associated with the mesh motion under consideration scales like $h^{r-1/2}$, whereas the jumps $\|\rho(t_+^n)-\rho(t^n)\|_{0,2,\Omega^{t^n}}$ scale like $h^{r+1/2}$, up to a logarithmic factor if $r=2$ [22]. This leads to an error estimate for universal meshes that is suboptimal by half an order in the L^2-norm, a result which is supported by the numerical examples in [21].

7. Conclusion. We have presented an a priori error analysis of finite element methods for problems with moving boundaries. We proved a general error estimate that applies to methods which employ a conforming mesh of the moving domain whose deformation is smooth in time except at a finite number of instants where remeshing is performed. Examples include ALE schemes with remeshing, as well as methods that employ a universal mesh as in [21]. Specializing the general error estimate to a given mesh motion strategy requires the estimation of certain quantities that depend on the mesh velocity v_h associated with the prescribed mesh motion. We illustrated such a calculation for an ALE scheme and intend to do the same for universal meshes in a separate paper.

We restricted our attention in this paper to deforming-mesh methods that adopt bijective mesh deformations to solve parabolic moving-boundary problems. This eliminated the need to handle variational crimes in the analysis, and it excluded the consideration of tangled meshes [10]. The analysis nonetheless provides a stepping stone toward the more difficult task of analyzing numerical solutions to free-boundary problems – problems for which the domain evolution is itself an unknown, such as phase-change problems, free-surface flows, and fluid-structure interaction. Needless to say, an analogous analysis of deforming-mesh methods for hyperbolic equations on moving domains is a topic worthy of further study as well.

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Appendix A. Auxiliary Lemmas.

Below, we prove Lemmas 5.3 and 6.1.

Proof. [Lemma 5.3] By the remark following (2.5), it suffices to check that the map $t \mapsto r_h^t u(\varphi_h^t(\cdot), t)$ belongs to $W^{1,1}(t^{n-1}, t^n, \mathcal{V}^{t^{n-1}})$. For this purpose, expand

$$r_h^t u = \sum_a \mathbf{r}_a(t) n_a^t$$

as a linear combination of the shape functions $n_a^t \in \mathcal{V}_h^t$ of Definition 2.3. According to (3.3), the coefficients $\mathbf{r}_a(t)$ satisfy the (nonsingular) linear system

$$\mathbf{A}(t)\mathbf{r}(t) = \mathbf{c}(t)$$

with

$$\mathbf{A}_{ab}(t) = a_h^t(n_b^t, n_a^t), \quad \mathbf{c}_a(t) = a_h^t(u, n_a^t).$$

The entries of $\mathbf{c}(t)$ are weakly differentiable by virtue of Lemma 5.2 and the regularity of u and n_a^t. On the other hand, the entries of $\mathbf{A}(t)$ are Lipschitz by the (temporally uniform) continuity of Λ_h^t. Furthermore, $\mathbf{A}(t)$ is uniformly positive definite by the (temporally uniform) coercivity of a_h^t. These facts can be used to show that the entries of $\mathbf{A}^{-1}(t)$ are Lipschitz. It then follows from the product rule for weak derivatives of univariate functions [8, Corollary 8.10] that the entries of $\mathbf{r}(t)$ are weakly differentiable, which proves the result. □

Proof. [Lemma 6.1] With $U = u \circ \varphi_h^t$ and $W = w \circ \varphi_h^t$, define extensions of u and w to all of $\Omega^{(t^{n-1}, t^n]}$ via $u(\cdot, \tau) = U \circ (\varphi_h^\tau)^{-1}$, $w(\cdot, \tau) = W \circ (\varphi_h^\tau)^{-1}$, $\tau \in (t^{n-1}, t^n]$. Then, by definition,

$$\Lambda_h^t(u, w) = \frac{d}{dt} \int_{\Omega^t} g \, dx$$

with

$$g = \nabla_x u \cdot \nabla_x w - (v_h \cdot \nabla_x u) w + \kappa u w.$$

Noting that the last three terms on the right-hand side of (6.2) coincide with $\int_{\Omega^t} g \nabla_x \cdot v_h \, dx$, the relation (6.2) will follow from Lemma 5.1 if we show that

$$D_t g = -\nabla_x u \cdot \left(\nabla_x v_h + (\nabla_x v_h)^\dagger \right) \nabla_x w - (\dot{v}_h \cdot \nabla_x u) w. \tag{A.1}$$

To this end, observe that the pullback $G(\cdot, t) := g(\varphi_h^t(\cdot), t)$ of g to $\Omega^{t^{n-1}}$ satisfies

$$G(\cdot, t) = \nabla_X U \cdot \left[(\nabla_X \varphi_h^t)^{-1} (\nabla_X \varphi_h^t)^{-\dagger} \right] \nabla_X W - \left((\nabla_X \varphi_h^t)^{-1} V_h \cdot \nabla_X U \right) W + \kappa U W,$$

where $V_h(\cdot, t) := v_h(\varphi_h^t(\cdot), t)$. Now differentiate with respect to time, using the fact that $\frac{\partial U}{\partial t} = \frac{\partial W}{\partial t} = 0$ together with the relations

$$\begin{aligned} \frac{\partial}{\partial t} \left[(\nabla_X \varphi_h^t)^{-1} (\nabla_X \varphi_h^t)^{-\dagger} \right] &= -(\nabla_X \varphi_h^t)^{-1} \left(K_h + (K_h)^\dagger \right) (\nabla_X \varphi_h^t)^{-\dagger}, \\ \frac{\partial}{\partial t} \left[(\nabla_X \varphi_h^t)^{-1} V_h \right] &= (\nabla_X \varphi_h^t)^{-1} \frac{\partial V_h}{\partial t} - (\nabla_X \varphi_h^t)^{-1} K_h V_h, \end{aligned}$$

where $K_h := \nabla_X V_h (\nabla_X \varphi_h^t)^{-1}$. Finally, recast the result on Ω^t to obtain (A.1). □