ZOLOTAREV’S FIFTH AND SIXTH PROBLEMS

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Abstract. In an influential 1877 paper, Zolotarev asked and answered four questions about polynomial and rational approximation. We ask and answer two questions: what are the best rational approximants $r$ and $s$ to $\sqrt{z}$ and $\text{sign}(z)$ on the unit circle (excluding certain arcs near the discontinuities), with the property that $|r(z)| = |s(z)| = 1$ for $|z| = 1$? We show that the solutions to these problems are related to Zolotarev’s third and fourth problems in a nontrivial manner.

1. Introduction

Nearly 150 years ago, Zolotarev asked and answered four questions from approximation theory [23]. The first two concern polynomial approximation. The third is equivalent to the fourth, and the fourth concerns the approximation of $\text{sign}(x) = x/\sqrt{x^2}$ by rational functions on $[-1, -\ell] \cup [\ell, 1]$, where $\ell \in (0, 1)$. His solutions to these problems have had lasting impact in approximation theory [1,9,19,21] and numerical analysis [2,6,10,11,13,16,20,22].

In this paper, we ask and answer two questions that are closely related to Zolotarev’s fourth problem: what are the best (in the uniform norm) rational approximants $r$ and $s$ to $\sqrt{z}$ and $\text{sign}(z) = z/\sqrt{z^2}$ on the unit circle (excluding certain arcs near the discontinuities), with the property that $|r(z)| = |s(z)| = 1$ for $|z| = 1$? We derive explicit solutions to these two problems and show that they are related in a nontrivial manner to the solution of Zolotarev’s fourth problem. We also show a remarkable property of these solutions: composing two best rational approximants of $\text{sign}(z)$ on the unit circle yields a best rational approximant of higher degree. This phenomenon closely mirrors the behavior of best rational approximants of $\text{sign}(x)$ on $[-1, -\ell] \cup [\ell, 1]$ [3,4,16]. Related composition laws for best rational approximants have appeared in other contexts, such as the approximation of the square root and $p$th root on positive real intervals [6–8] and the solution of certain extremal problems involving finite Blaschke products [17, 18]. Some other rational approximation problems on the unit circle have been studied in [14,15].

Let us give precise statements of the problems that we study, beginning with some notation. We say that a rational function $r(z) = p(z)/q(z)$ has type $(m, n)$ if $p$ and $q$ are polynomials of degree at most $m$ and $n$, respectively. We use $\mathcal{R}_{m,n}$ to denote the set of rational functions of type $(m, n)$ with complex coefficients, and $\mathcal{R}_{\text{real}}^{m,n}$ to denote the set of rational functions of type $(m, n)$ with real coefficients. We say that $r \in \mathcal{R}_{m,n}$ has exact type $(\mu, \nu)$ if, after canceling common factors, $p$
and \( q \) have degree exactly \( \mu \) and \( \nu \), respectively. For each \( \Theta \in (0, \pi/2) \), we let

\[
S_\Theta = \{ e^{i\theta} : \theta \in [-2\Theta, 2\Theta] \},
\]

\[
T_\Theta = \{ e^{i\theta} : \theta \in [-\Theta, \Theta] \cup [\pi - \Theta, \pi + \Theta] \}.
\]

We address the following rational approximation problems.

**Problem Z5.** Given \( \Theta \in (0, \pi/2) \) and \( n \in \mathbb{N}_0 \), find the rational function in \( \{ r \in \mathcal{R}_{n,n} : |r(z)| = 1 \text{ on } |z| = 1 \} \) that minimizes

\[
\max_{z \in S_\Theta} \left| \arg \left( \frac{r(z)}{\sqrt{z}} \right) \right|.
\]

**Problem Z6.** Given \( \Theta \in (0, \pi/2) \) and \( m \in \mathbb{N}_0 \), find the rational function in \( \{ r \in \mathcal{R}_{m,m} : |r(z)| = 1 \text{ on } |z| = 1 \} \) that minimizes

\[
\max_{z \in T_\Theta} \left| \arg \left( \frac{r(z \, \text{sign}(z))}{\sqrt{z}} \right) \right|.
\]

We labeled the above problems “Z5” and “Z6” since they are natural follow-ups to Zolotarev’s fourth problem, which we label “Z4”. Zolotarev’s fourth problem reads as follows (up to a trivial scaling).

**Problem Z4.** Given \( \ell \in (0, 1) \) and \( m \in \mathbb{N}_0 \), find the rational function \( r \in \mathcal{R}_{m,m}^{\text{real}} \) that minimizes

\[
\max_{x \in [-1, -\ell] \cup [\ell, 1]} |r(x) - \text{sign}(x)|.
\]

We will derive explicit solutions to Problems Z5 and Z6 in Section 2. As we shall see, the solutions to problems Z4–Z6 are connected in a nontrivial manner. Then we will study their properties in Section 3, including their behavior under composition and their error.

### 2. Solutions

In this section, we derive explicit solutions to Problems Z5 and Z6. The solutions, summarized in Theorem 2.1, involve Jacobi’s elliptic functions. We use \( \text{sn}(\cdot, \ell) \), \( \text{cn}(\cdot, \ell) \), and \( \text{dn}(\cdot, \ell) \) to denote Jacobi’s elliptic functions with modulus \( \ell \), and we use \( \ell' = \sqrt{1 - \ell^2} \) to denote the modulus complementary to \( \ell \). We denote the complete elliptic integral of the first kind by \( K(\ell) = \int_0^{\pi/2} (1 - \ell^2 \sin^2 \theta)^{-1/2} d\theta \).

**Theorem 2.1.** Let \( m, n \in \mathbb{N}_0 \) and \( \Theta \in (0, \pi/2) \). Problem Z5 has a unique solution given by

\[
(2.1) \quad r(z) = r_n(z; \Theta) = \prod_{j=1}^n \frac{1 + a_j z}{z + a_j},
\]

where

\[
(2.2) \quad a_j = \left( \frac{\ell \text{sn} \left( \frac{2j-1}{2n+1} K(\ell'), \ell' \right) + \text{dn} \left( \frac{2j-1}{2n+1} K(\ell'), \ell' \right)}{\text{cn} \left( \frac{2j-1}{2n+1} K(\ell'), \ell' \right)} \right)^{2(-1)^{j+n}},
\]

and \( \ell = \cos \Theta \). Problem Z6 has two solutions: the function

\[
(2.3) \quad s(z) = s_m(z; \Theta) = i^{1-m} \prod_{j=1}^m \frac{z - ib_j}{1 + ib_j z},
\]
and its reciprocal, where

\begin{equation}
(2.4) \quad b_j = (-1)^{m_j} \left( \frac{\ell \text{sn}(\frac{2j-1}{m}K(\ell'), \ell') + \text{dn}(\frac{2j-1}{m}K(\ell'), \ell')}{\text{cn}(\frac{2j-1}{m}K(\ell'), \ell')} \right)^{-1/j}.
\end{equation}

Remark 2.2. When \(m = 2n + 1\), the functions \(s_m(z; \Theta)\) and \(r_n(z; \Theta)\) are related to one another. Using the observation that

\[
K = \text{determined uniquely by the condition that} \quad K = \text{determined uniquely by the condition that} \quad K
\]

one checks that

\begin{equation}
(2.5) \quad s_{2n+1}(z; \Theta)^{-1/n} = z \prod_{j=1}^{n} \left( \frac{z^2 + a_j}{1 + a_j z^2} \right) = \frac{z}{r_n(z^2; \Theta)}.
\end{equation}

In particular, \(s_{2n+1}(z; \Theta)\) has exact type \((2n+1, 2n)\) when \(n\) is even, and it has exact type \((2n, 2n+1)\) when \(n\) is odd. On the other hand, \(s_{2n}(z; \Theta)\) has exact type \((2n, 2n)\).

Remark 2.3. For \(m, n > 1\), neither \(s_m(z; \Theta)\) nor \(r_n(z; \Theta)\) is a finite Blaschke product, since both functions have at least one root outside the unit disk.

The following identity will play a central role in our proof of Theorem 2.1.

Theorem 2.4. Let \(\ell \in (0, 1)\) and \(m \in \mathbb{N}\). Let \(M = K(\ell)/K(\lambda)\), where \(\lambda \in (0, 1)\) is determined uniquely by the condition that

\begin{equation}
(2.6) \quad \frac{K(\ell)}{K(\ell')} = \frac{K(\lambda)}{mK(\lambda')}
\end{equation}

holds with \(\lambda' = \sqrt{1 - \lambda^2}\). Then the function \(s(z)\) in (2.3) can be expressed as

\begin{equation}
(2.7) \quad s(z) = F(x) + i \text{sign}(\text{Im} z)^m G(x), \quad x = \frac{1}{2}(z + z^{-1}),
\end{equation}

where

\begin{equation}
(2.8) \quad F(x) = F_m(x; \ell) = \lambda \text{sn} \left( \frac{\text{sn}^{-1}(x/\ell, \ell)}{M}, \lambda \right),
\end{equation}

\begin{equation}
(2.9) \quad G(x) = G_m(x; \ell) = \text{dn} \left( \frac{\text{sn}^{-1}(x/\ell, \ell)}{M}, \lambda \right).
\end{equation}

The function \(F(x)\) appearing above is none other than Zolotarev’s classical solution to Problem Z4 on \([-1, -\ell] \cup [\ell, 1]\), scaled to have maximum value 1 on \([\ell, 1]\) ([23], [1, Sections 50-51]):

\[
\frac{2}{1 + \lambda} F = \arg \min_{r \in \mathbb{R}_{\text{real}}} \max_{x \in [-1, -\ell] \cup [\ell, 1]} |r(x) - \text{sign}(x)|.
\]

It is well-known that \(F(x)\) is an odd rational function of exact type \((2[(m-1)/2] + 1, 2(m/2))\) that is real-valued on \(\mathbb{R}\) and oscillates between \(\lambda\) and 1 on \([\ell, 1] = [\cos \Theta, 1]\), achieving these values at \(m + 1\) points \(\ell = x_0 < x_1 < \cdots < x_m = 1\) in an
alternating fashion ([2, p. 9], [1, Sections 50-51]). In particular, \( F(\ell) = \lambda \). Since \(|s(z)| = 1\) for \(|z| = 1\), it follows from (2.7) that \( \arg(s(e^{i\theta})) \) equioscillates \( m + 1 \) times on \([-\Theta, \Theta]\), taking values in \([-\arccos \lambda, \arccos \lambda]\). That is,

\[
\arg(s(e^{i\theta})) = (-1)^j \max_{\theta \in [-\Theta, \Theta]} |\arg(s(e^{i\theta}))| = (-1)^j \arccos \lambda, \quad j = 0, 1, \ldots, m,
\]

where \( \sigma \in \{-1, 1\} \) and

\[
\theta_j = \begin{cases} 
-\arccos x_{2j}, & \text{if } j \leq m/2, \\
\arccos x_{2m-2j}, & \text{if } j > m/2.
\end{cases}
\]

We will eventually use this fact, together with Remark 2.2, to prove the optimality of \( s \) and \( r \).

### 2.1. Proof of Theorem 2.4

Let us first prove Theorem 2.4, beginning with the case in which \( m = 2n + 1 \) and \( n \) is even.

**Case 1** \((m = 2n + 1, n \text{ even})\). The fact that the right-hand side of (2.7) is a rational function of \( z \), much less of type \((2n + 1, 2n)\), is not obvious at first glance. To prove this, we recall the identities [1, p. 214]

\[
\text{(2.11)} \quad \sn \left( \frac{u}{M}, \lambda \right) = \frac{\sn(u, \ell)}{M} \prod_{k=1}^{n} \frac{1 + \sn^2(u, \ell)}{1 + \sn^2(u, \ell) \frac{\cn^2(v_{2k-1}, \ell')}{\dn^2(v_{2k-1}, \ell')}},
\]

\[
\text{(2.12)} \quad \dn \left( \frac{u}{M}, \lambda \right) = \frac{\dn(u, \ell)}{M} \prod_{k=1}^{n} \frac{1 - \sn^2(u, \ell)}{1 + \sn^2(u, \ell) \frac{\cn^2(v_{2k-1}, \ell')}{\dn^2(v_{2k-1}, \ell')}},
\]

where \( v_j = \frac{1}{m} K(\ell') \). Let us denote

\[
\text{(2.13)} \quad \tilde{F}(z) = \tilde{F}_{2n+1}(z; \Theta) = F_{2n+1} \left( \frac{1}{2} (z + z^{-1}); \ell \right),
\]

\[
\text{(2.14)} \quad \tilde{G}(z) = \tilde{G}_{2n+1}(z; \Theta) = \text{sign}(\text{Im } z) G_{2n+1} \left( \frac{1}{2} (z + z^{-1}); \ell \right).
\]

Note that \( \tilde{F}(z) - i \tilde{G}(z) = (\tilde{F}(z) + i \tilde{G}(z))^{-1} \) since \( \lambda^2 \sn^2(\cdot, \lambda) + \dn^2(\cdot, \lambda) = 1 \). Using the fact that

\[
\ell \sn(u, \ell) = \frac{1}{2} (z + z^{-1}) \iff \dn(u, \ell) = \frac{1}{2\ell} (z - z^{-1}) \text{ sign(Im } z),
\]

we can write

\[
\text{From these expressions it is easy to deduce that } \tilde{F}(z) + i \tilde{G}(z) \text{ is a rational function which is ostensibly of type } (4n + 2, 4n + 1). \text{ However, this turns out to be an overestimate: } \tilde{F}(z) \text{ and } i \tilde{G}(z) \text{ have } 2n + 1 \text{ coincident poles (one of which is at } z = 0) \text{ with opposite residues, rendering } \tilde{F}(z) + i \tilde{G}(z) \text{ of type } (2n + 1, 2n).
To see why, it is helpful to rewrite $F(x)$ and $G(x)$ in terms of the Grösch ring function
$$
\mu(\lambda) = \frac{\pi K(\lambda')}{2 K(\lambda)}, \quad \lambda' = \sqrt{1 - \lambda^2}
$$
and the functions
$$
f_{\nu}(x) = \ell \text{sn}(K(\ell)x, \ell), \quad g_{\nu}(x) = \text{dn}(K(\ell)x, \ell), \quad \ell = \mu^{-1}(1/\nu).
$$

One readily checks, using (2.6), that
$$
F(x) = f_{m\nu}(f_{\nu}^{-1}(x)), \quad \nu = \frac{1}{\mu(\ell)}.
$$

Similar formulas for $F$ appear in [3,4].

Next, we recall that the poles of $\sn(u, \lambda)$ occur at $u \in \{2pK(\lambda) + i(2j - 1)K(\lambda') \mid p, j \in \mathbb{Z}\}$ [12, Equation 2.2.9]. The finite nonzero poles of $\tilde{F}(z)$ thus occur at those $z \in \mathbb{C}$ for which
$$
K(\lambda) f_{\nu}^{-1} \left( \frac{1}{2} (z + z^{-1}) \right) = 2pK(\lambda) + i(2j - 1)K(\lambda'), \quad p, j \in \mathbb{Z}.
$$

That is,
$$
\frac{1}{2} (z + z^{-1}) = f_{\nu} \left( 2p + i(2j - 1) \frac{K(\lambda')}{K(\lambda)} \right)
= f_{\nu} \left( 2p + i \frac{2j - 1}{m} \frac{K(\ell')}{K(\ell)} \right)
= \ell \text{sn} \left( 2pK(\ell) + i \frac{2j - 1}{m} K(\ell'), \ell \right)
= (-1)^p \ell \text{sn}(iv_{2j-1}, \ell).
$$

Here, we used (2.6), the notation $v_j = \frac{1}{m} K(\ell')$, and the half-period identity $\text{sn}(2pK(\ell) + u, \ell) = (-1)^p \text{sn}(u, \ell)$ [12, Equation 2.2.11].

The numbers $z$ satisfying $\frac{1}{2} (z + z^{-1}) = (-1)^p \ell \text{sn}(iv_{2j-1}, \ell)$ are given by
$$
z = (-1)^p \ell \text{sn}(iv_{2j-1}, \ell) \pm i \text{dn}(iv_{2j-1}, \ell).
$$

Indeed, since $\ell^2 \text{sn}^2(\cdot, \ell) + \text{dn}^2(\cdot, \ell) = 1$, we have $z^{-1} = (-1)^p \ell \text{sn}(iv_{2j-1}, \ell) \mp i \text{dn}(iv_{2j-1}, \ell)$.

We conclude that the finite nonzero poles of $\tilde{F}(z)$ occur at
$$
\{z_{j,p,q} \mid p, q = 0, 1, j = 1, 2, \ldots, n\},
$$
where
$$
z_{j,p,q} = (-1)^p (\ell \text{sn}(iv_{2j-1}, \ell) + (-1)^q i \text{dn}(iv_{2j-1}, \ell)).
$$

The finite nonzero poles of $i\tilde{G}(z)$ are identical, since $\text{dn}(\cdot, \lambda)$ and $\text{sn}(\cdot, \lambda)$ have the same poles. All of these poles are simple poles thanks to the simplicity of the poles of $\sn$ and $\dn$.

Below we relate the residues of $\tilde{F}(z)$ to those of $i\tilde{G}(z)$.

**Lemma 2.5.** We have
$$
\text{Res}(\tilde{F}, z_{j,p,q}) = \begin{cases} 
\text{Res}(i\tilde{G}, z_{j,p,q}), & \text{if } j + q \text{ is odd}, \\
-\text{Res}(i\tilde{G}, z_{j,p,q}), & \text{if } j + q \text{ is even}.
\end{cases}
$$

In particular, $\text{Res}(\tilde{F} + i\tilde{G}, z_{j,p,q}) = 0$ if $j + q$ is even.
\textbf{Proof.} In view of (2.15), the residues of $F\left(\frac{1}{2}(z + z^{-1})\right)$ and $G\left(\frac{1}{2}(z + z^{-1})\right)$ at $z_{j,p,q}$ are proportional to the residues of $f_{ju}(u) = \lambda \text{sn}(K(u), \lambda)$ and $g_{ju}(u) = \text{dn}(K(u), \lambda)$ at $u = f^{-1}\nu\left(\frac{1}{2}(z_{j,p,q} + z_{j,p,q}^{-1})\right) =: u_{j,p,q}$, with the constant of proportionality the same in both cases. From (2.16), we have

$$K(u_{j,p,q}) = 2pK(\lambda) + i(2j - 1)K(\lambda'),$$

so [12, p. 41-42]

(2.18) \hspace{1cm} \text{Res}(\lambda \text{sn}(K(u)\lambda), u_{j,p,q}) = (-1)^p/K(\lambda),

(2.19) \hspace{1cm} \text{Res}(\text{dn}(K(u)\lambda), u_{j,p,q}) = (-1)i/K(\lambda).

Since $\text{sign}(\text{Im} z_{j,p,q}) = (-1)^{p+q}$, it follows that

(2.20) \hspace{1cm} i \text{sign}(\text{Im} z_{j,p,q}) \text{Res}(\text{dn}(K(u)\lambda), u_{j,p,q}) = (-1)^{j+q+1}/K(\lambda).

Comparing (2.18) with (2.20), we see that the residues of $\tilde{F}$ and $i\tilde{G}$ are equal if $j + q$ is odd, and they are opposite if $j + q$ is even. \hfill \Box

We conclude that the function $\tilde{F}(z) + i\tilde{G}(z)$ has only $2n$ finite nonzero poles,

$$\pm (\ell \text{sn}(iv_{2j-1}, \ell) + (-1)^{j+1}i \text{dn}(iv_{2j-1}, \ell)), \quad j = 1, 2, \ldots, n.$$

All of these poles are simple. Since $\tilde{F}(z) + i\tilde{G}(z)$ has unit modulus on the unit circle, its finite nonzero roots are the reciprocals of these poles.

Now observe that since $\text{sn}(iu, \ell) = i^\ell \text{cn}(u, \ell')$ and $\text{dn}(iu, \ell) = i^\ell \text{cn}(u, \ell')$ [12, Equation 2.6.12], we have

$$\left(\ell \text{sn}(iv_{2j-1}, \ell) + (-1)^{j+1}i \text{dn}(iv_{2j-1}, \ell)\right)^2 = \left(\ell \text{sn}(iv_{2j-1}, \ell) + i \text{dn}(iv_{2j-1}, \ell)\right)^{2(-1)^{j+1}}$$

$$= \left(\frac{i\ell \text{sn}(v_{2j-1}, \ell')} + i \text{dn}(v_{2j-1}, \ell')\right)^{2{-1)^{j+1}}$$

$$= \left(\frac{\ell \text{sn}(v_{2j-1}, \ell') + \text{dn}(v_{2j-1}, \ell')}{\text{cn}(v_{2j-1}, \ell')}\right)^{2{-1)^{j+1}}$$

$$= -1/a_j,$$

(2.21)

where $a_j$ is given by (2.2) (recall that we are still focusing on the case in which $m = 2n + 1$ and $n$ is even).

It follows that

(2.22) \hspace{1cm} \tilde{F}(z) + i\tilde{G}(z) = e^{i\alpha}z^k \prod_{j=1}^n \frac{z^2 + a_j}{1 + a_j z^2}

for some $\alpha \in \mathbb{R}$ and some $k \in \mathbb{Z}$. We must have $e^{i\alpha} = 1$ since $\tilde{F}(1) = F(1) > 0$ and $\tilde{G}(1) = 0$. We must have $k \geq -1$ since $\tilde{F}(z)$ and $\tilde{G}(z)$ each have simple poles at $z = 0$. We must have $k \leq 1$ for a similar reason: $\tilde{F}(z) - i\tilde{G}(z) = \frac{1}{F(z) + iG(z)}$ cannot have a pole of order $> 1$ at $z = 0$. To conclude, note that at $z = i$, the left-hand side of (2.22) evaluates to $i$, while the right-hand side evaluates to $i^k(-1)^n = i^k$. The only possibility is $k = 1$. Thus,

$$\tilde{F}(z) + i\tilde{G}(z) = z \prod_{j=1}^n \frac{z^2 + a_j}{1 + a_j z^2} = \frac{z}{r_n(z^2; \Theta)}, \quad \text{if } m = 2n + 1 \text{ and } n \text{ is even.}$$
In view of (2.5), this completes the proof of Theorem 2.4 for the case in which
\( m = 2n + 1 \) and \( n \) is even.

**Case 2** \((m = 2n + 1, n \text{ odd})\). The case in which \( m = 2n + 1 \) and \( n \) is odd is handled similarly. This time, (2.21) becomes

\[
\left( \ell \operatorname{sn}(iv_{2j-1}, \ell) + (-1)^{j+1}i \operatorname{dn}(iv_{2j-1}, \ell) \right)^2 = -a_j,
\]

so that (2.22) becomes

\[
(2.23) \quad \tilde{F}(z) + i\tilde{G}(z) = e^{i\alpha}z^k \prod_{j=1}^{n} \frac{1 + a_j z^2}{z^2 + a_j}.
\]

As before, we can argue that \( e^{i\alpha} = 1 \) and \(-1 \leq k \leq 1\). At \( z = i \), the left-hand side evaluates to \( i \), while the right-hand side evaluates to \( i^k(-1)^n = -i^k \). We conclude that \( k = -1 \). That is,

\[
\tilde{F}(z) + i\tilde{G}(z) = \frac{1}{z} \prod_{j=1}^{n} \frac{1 + a_j z^2}{z^2 + a_j} = \frac{r_n(z^2; \Theta)}{z}, \quad \text{if} \ m = 2n + 1 \text{ and } n \text{ is odd.}
\]

**Case 3** \((m = 2n)\). Finally, when \( m = 2n \), the identities (2.11-2.12) change to [1, p. 214]

\[
(2.24) \quad \operatorname{sn} \left( \frac{u}{M}, \lambda \right) = \frac{\operatorname{sn}(u, \ell)}{M} \prod_{k=1}^{n-1} \frac{1 + \operatorname{sn}^2(u, \ell) \frac{\operatorname{cn}^2(v_{2k-1}, \ell')}{\operatorname{sn}^2(v_{2k-1}, \ell')}}{1 + \operatorname{sn}^2(u, \ell) \frac{\operatorname{cn}^2(v_{2k-1}, \ell')}{\operatorname{sn}^2(v_{2k-1}, \ell')}},
\]

\[
(2.25) \quad \operatorname{dn} \left( \frac{u}{M}, \lambda \right) = \prod_{k=1}^{n} \frac{1 - \operatorname{sn}^2(u, \ell) \frac{\operatorname{dn}^2(v_{2k-1}, \ell')}{\operatorname{sn}^2(v_{2k-1}, \ell')}}{1 + \operatorname{sn}^2(u, \ell) \frac{\operatorname{dn}^2(v_{2k-1}, \ell')}{\operatorname{sn}^2(v_{2k-1}, \ell')}}.
\]

Note that in contrast to [1, p. 214], we terminated the product in the numerator
of (2.24) at \( k = n - 1 \) rather than \( k = n \) since \( \operatorname{cn}(v_{2n}, \ell') = 0 \) when \( m = 2n \). Accordingly, we put

\[
(2.26) \quad \tilde{F}(z) = \tilde{F}_{2n}(z; \Theta) = F_{2n} \left( \frac{1}{2}(z + z^{-1}); \ell \right),
\]

\[
(2.27) \quad \tilde{G}(z) = \tilde{G}_{2n}(z; \Theta) = G_{2n} \left( \frac{1}{2}(z + z^{-1}); \ell \right),
\]

and we observe that \( \tilde{F}(z) + i\tilde{G}(z) \) is a rational function which is ostensibly of type
\((4n, 4n)\). However, \( 2n \) of the poles \( z_{j,p,q} \) coincide and have opposite residues; this
\( \ell \) is those poles \( z_{j,p,q} \) for which \( j + p \) is even, since the factor \( \operatorname{sign}(\operatorname{Im}z_{j,p,q}) \)
does not appear in the analysis (compare (2.27) with (2.14)). Since \( \tilde{F} \) and \( i\tilde{G} \) have
\( 2n \) coincident poles with opposite residues, \( \tilde{F}(z) + i\tilde{G}(z) \) is in fact of type \((2n, 2n)\).

The poles of \( \hat{F}(z) + i\hat{G}(z) \) are

\[
(2.28) \quad (-1)^{j+1} \left( \ell \operatorname{sn}(iv_{2j-1}, \ell) \pm i \operatorname{dn}(iv_{2j-1}, \ell) \right) = \pm (-1)^{j+1}ib_j \quad \text{or} \quad (-ib_j) \quad \text{for} \quad j = 1, 2, \ldots, n,
\]
where \( b_j \) is given by (2.4). One checks that the sets \( \{-(ib_j)^{(-1)^j}\}_{j=1}^n \cup \{-ib_j^{-(1)^j}\}_{j=1}^n \) and \( \{-1/(ib_j)\}_{j=1}^m \) are equal, so \( \tilde{F}(z) + i\tilde{G}(z) \) must have the form

\[
\tilde{F}(z) + i\tilde{G}(z) = e^{i\alpha}z^k \prod_{j=1}^m \frac{z - ib_j}{1 + ib_jz}.
\]

Again, we can argue that \(-1 \leq k \leq 1\), but this time we cannot conclude that \( e^{i\alpha} = 1 \) by evaluating both sides of (2.28) at \( z = 1 \). Instead, we evaluate both sides at \( z = i \) to obtain \( i = e^{i\alpha}k + m \), and we evaluate both sides at \( z = -i \) to obtain \( i = e^{i\alpha}(-i)^{k+m} \). We conclude that \( k + m \) is even, and since \( m \) is too, we have \( k = 0 \) and \( e^{i\alpha} = i^{1-m} \).

This completes the proof of Theorem 2.4.

As a final remark, we note that Theorem 2.4 also holds trivially when \( m = 0 \) if we adopt the convention that \( \lambda := 0 \) when \( m = 0 \).

2.2. Proof of Theorem 2.1. Let us now use Theorem 2.4 to prove Theorem 2.1. We first elaborate on the relation between Problems Z5 and Z6. Observe that if

\[
\text{arg}(zp(z^2)/q(z^2)) = \text{arg}(q(w)/p(w)) = \text{arg}(q(z^2)/(zp(z^2)))/\text{sign}(z)
\]

for any polynomials \( p \) and \( q \). In view of (2.10) and Remark 2.2, it follows that \( \text{arg}(r(e^{i\theta})/\sqrt{e^{i\theta}}) \) equioscillates \( 2n+2 \) times on \([-\Theta, 2\Theta]\), taking values in \([-\arccos \lambda, \arccos \lambda]\). Suppose now that \( \bar{r}(z) \) is another rational function of type \((n, n)\) satisfying \(|\bar{r}(z)| = 1\) for \(|z| = 1\) and

\[
\max_{z \in \mathcal{S}_\Theta} \left| \text{arg} \left( \frac{\bar{r}(z)}{\sqrt{z}} \right) \right| \leq \arccos \lambda.
\]

Then the equioscillation of \( \text{arg}(r(e^{i\theta})/\sqrt{e^{i\theta}}) \) implies that on \([-2\Theta, 2\Theta]\),

\[
\text{arg} \left( \frac{r(e^{i\theta})}{\sqrt{e^{i\theta}}} \right) - \text{arg} \left( \frac{\bar{r}(e^{i\theta})}{\sqrt{e^{i\theta}}} \right) = \text{arg} \left( \frac{r(e^{i\theta})}{\bar{r}(e^{i\theta})} \right)
\]

has at least \( 2n+1 \) roots, counted with multiplicity. Hence, the numerator of \( r(z) - \bar{r}(z) \) has at least \( 2n+1 \) roots, counted with multiplicity. Since \( r(z) - \bar{r}(z) \) has type \((2n, 2n)\), it follows that \( r = \bar{r} \). This shows that Problem Z5 has a unique solution, namely \( r \).

The proof that Problem Z6 has precisely two solutions—\( s(z) \) and \( s(z)^{-1} \)—proceeds similarly. Assume \( m > 0 \); otherwise the claim is trivial. We see from (2.10) that

\[
\max_{z \in \mathcal{T}_\Theta} \left| \text{arg} \left( \frac{s(z)}{\text{sign}(e^{i\theta})} \right) \right| = \arccos \lambda,
\]

and \( \text{arg}(s(e^{i\theta})/\text{sign}(e^{i\theta})) \) equioscillates \( m+1 \) times on \([\Theta, \Theta]\) and \( m+1 \) times on \([\pi - \Theta, \pi + \Theta]\), owing to the fact that \( \text{sign}(e^{i\theta}) = 1 \) when \( \theta \in [\Theta, \Theta] \), \( \text{sign}(e^{i\theta}) = -1 \) when \( \theta \in [\pi - \Theta, \pi + \Theta] \), and \( -s(e^{i\theta}) = s(e^{i(\pi-\theta)})^{-1} \) for all \( \theta \). The same is true of \( \text{sign}(e^{i\theta})^{-1} \)

\[
\text{arg} \left( \frac{s(e^{i\theta})^{-1}}{\text{sign}(e^{i\theta})} \right) = -\text{arg} \left( \frac{s(e^{i\theta})}{\text{sign}(e^{i\theta})} \right)
\]
for all $\theta$. Suppose now that $\tilde{s}(z)$ is another rational function of type $(m, m)$ satisfying $|\tilde{s}(z)| = 1$ for $|z| = 1$ and

$$\max_{z \in T_{\Theta}} \left| \arg \left( \frac{\tilde{s}(z)}{\operatorname{sign} z} \right) \right| \leq \arccos \lambda.$$  

Then the same reasoning as above shows that the numerator of $\tilde{s}(z) - s(z)$ has at least $2m$ roots counted with multiplicity. At least $m$ of them lie in $\{ z \in T_{\Theta} \mid \Re z > 0 \}$, and at least $m$ of them lie in $\{ z \in T_{\Theta} \mid \Re z < 0 \}$. Likewise, the numerator of $\tilde{s}(z) - s(z)^{-1}$ has at least $2m$ roots counted with multiplicity, at least $m$ of which lie in $\{ z \in T_{\Theta} \mid \Re z > 0 \}$ and at least $m$ of which lie in $\{ z \in T_{\Theta} \mid \Re z < 0 \}$. By considering the graphs of $\arg s(z)$ and $\arg(s(z)^{-1})$ (see Figure 1), there must also be at least one point $z \in \{ z \in \mathbb{C} \mid |z| = 1, z \notin \text{int}(T_{\Theta}) \}$ where either $\tilde{s}(z) = s(z)$ or $\tilde{s}(z) = s(z)^{-1}$. (If all such points happen to be on the boundary of $T_{\Theta}$, then it is easy to see that there must have been more than $2m$ points in $T_{\Theta}$ (counting multiplicities) where either $\tilde{s}(z) = s(z)$ or $\tilde{s}(z) = s(z)^{-1}$ to begin with.) We conclude that either $\tilde{s} = s$ or $\tilde{s} = 1/s$. This shows that Problem Z6 has precisely two solutions: $s$ and $1/s$.

**3. Properties of the Solutions**

In this section, we study the error committed by the functions $r_n(z; \Theta)$ and $s_m(z; \Theta)$ from Theorem 2.1, and we study the behavior of $r_n(z; \Theta)$ and $s_m(z; \Theta)$ under composition.

**3.1. Error.** To study the error, we appeal to well-known properties of the function $F_m(x; \ell)$ defined in (2.8). As we noted earlier, $\frac{2}{1+\lambda} F_m(x; \ell)$ is the solution to Problem Z4 on $[-1, -\ell] \cup [\ell, 1]$. 

**Figure 1.** Plots of $\arg s_m(z; \Theta)$ with $\Theta = \pi/2 - 0.1$ and $m = 4, 5$. Portions of the graph corresponding to points $z \in T_{\Theta}$ (respectively, $z \notin T_{\Theta}$) are colored blue (respectively, red). Extrema of the error on $T_{\Theta}$ are marked with blue dots. The dashed line is $\arg \operatorname{sign} z$. Both the horizontal and vertical axes are to be interpreted modulo $2\pi$. The graphs of $\arg(s_m(z; \Theta)^{-1})$ are obtained by reflecting the above graphs across the horizontal axis.
The number \( \frac{1-\lambda}{1+\lambda} = \max_{x \in [-\ell, \ell]} \left| \frac{2}{1+\lambda} F_m(x; \ell) - \text{sign}(x) \right| \) is well-studied; it satisfies \([2, \text{p. 9}]\)

\[
(3.1) \quad \frac{1-\lambda}{1+\lambda} = \frac{2\sqrt{Z_m}}{1+Z_m},
\]

where \( Z_m = Z_m([-1, -\ell], [\ell, 1]) \) denotes the Zolotarev number of the sets \([-1, -\ell] \) and \([\ell, 1] \):

\[
(3.2) \quad Z_m(E, F) = \inf_{r \in \mathbb{R}} \sup_{x \in E} |r(x)| \inf_{z \in F} |r(z)|.
\]

An explicit formula for \( Z_m \) \((m \geq 1)\) is \([2, \text{Theorem 3.1}]\)

\[
Z_m = 4\rho^{-2m} \prod_{j=1}^{\infty} \frac{(1+\rho^{-8jm})^4}{(1+\rho^{jm}\rho^{-8jm})^4} \leq 4\rho^{-2m},
\]

where

\[
\rho = \exp \left( \frac{\pi K(\ell)}{K'(\ell')} \right) = \exp \left( \frac{\pi K(\cos \Theta)}{K(\sin \Theta)} \right).
\]

Note that the bound \( Z_m \leq 4\rho^{-2m} \) also obviously holds for \( m = 0 \). Solving for \( \lambda \) in \((3.1)\), we find that

\[
(3.3) \quad \max_{z \in \mathbb{T}_m} \left| \text{arg} \left( \frac{s_m(z; \Theta)}{\text{sign}(z)} \right) \right| = \arccos \lambda = \arccos \left( \frac{1-\sqrt{Z_m}}{1+\sqrt{Z_m}} \right).
\]

We derive upper bounds for this quantity below.

**Lemma 3.1.** For every \( x \geq 0 \),

\[
\arccos \left( \frac{1-\sqrt{x}}{1+\sqrt{x}} \right) \leq 2\sqrt{2\pi^{1/4}}.
\]

**Proof.** Let \( f(x) = \arccos \left( \frac{1-\sqrt{x}}{1+\sqrt{x}} \right) \) and \( g(x) = 2\sqrt{2\pi^{1/4}}. \) Since \( f(0) = g(0) = 0 \) and

\[
f'(x) = \frac{1 - \sqrt{x}}{\sqrt{2\pi^{3/4}(1+\sqrt{x})}\sqrt{1+x}} < \frac{1}{\sqrt{2\pi^{3/4}}} = g'(x), \quad x > 0,
\]

we have \( f(x) = \int_0^x f'(t) \, dt \leq \int_0^x g'(t) \, dt = g(x) \) for every \( x \geq 0 \). \( \square \)

**Theorem 3.2.** Let \( \Theta \in (0, \pi/2) \) and \( m, n \in \mathbb{N}_0 \). We have

\[
(3.4) \quad \max_{z \in \mathbb{T}_m} \left| \text{arg} \left( \frac{s_m(z; \Theta)}{\text{sign}(z)} \right) \right| \leq 4\rho^{-m/2} \leq 4 \left[ \exp \left( \frac{\pi^2}{4\log(4 \sec \Theta)} \right) \right]^{-m}
\]

and

\[
(3.5) \quad \max_{z \in \mathbb{S}_n} \left| \text{arg} \left( \frac{r_n(z; \Theta)}{\sqrt{z}} \right) \right| \leq 4\rho^{-(n+1)/2} \leq 4 \left[ \exp \left( \frac{\pi^2}{2\log(4 \sec \Theta)} \right) \right]^{-(n+1/2)}.
\]

**Proof.** Using Lemma 3.1 and the inequality \([2, \text{p. 8}]\)

\[
\frac{\pi}{2} K(\sqrt{1 - x^2}) / K(x) \leq \log 4/x, \quad 0 < x < 1,
\]

\[
\frac{\pi}{2} K(\sqrt{1 - x^2}) / K(x) = \log 4/x, \quad 0 < x < 1,
\]
we compute
\[
\max_{z \in T_0} \left| \arg \left( \frac{s_m(z; \Theta)}{\text{sign}(z)} \right) \right| \leq 2 \sqrt{2} Z_{1/4} \leq 2 \sqrt{2} (4 \rho^{-2m})^{1/4} = 4 \rho^{-m/2} \leq 4 \left[ \exp \left( \frac{\pi^2}{2 \log(4 \sec \Theta)} \right) \right]^{-m/2}.
\]
The bound (3.5) follows from Remark 2.2 and (2.29), which imply
\[
\max_{z \in S_0} \left| \arg \left( \frac{r_{m+1}(z; \Theta)}{\sqrt{z}} \right) \right| = \max_{z \in T_0} \left| \arg \left( \frac{s_2(z; \Theta)}{\text{sign}(z)} \right) \right|.
\]

Theorem 3.2 is illustrated in Figure 2, which shows the bounds are very tight.

Figure 3 plots the absolute errors $|r_{n}(z; \Theta) - \sqrt{z}|$ and $|s_{m}(z; \Theta) - \text{sign}(z)|$ for $z \in \mathbb{C}$.

### 3.2. Composition.

Next, we show that when two solutions of Problem Z6 are composed with one another, the resulting function is a solution of Problem Z6 of higher degree.

**Theorem 3.3.** Let $\Theta \in (0, \pi/2)$, $m, \bar{m} \in \mathbb{N}_0$, and $\bar{\Theta} = \arg(s_m(e^{i\Theta}; \Theta))$. Then
\[
s_{\bar{m}}(s_m(z; \Theta); \bar{\Theta}) = s_{\bar{m}m}(z; \Theta).
\]

**Proof.** This is essentially a consequence of the identities
\[
f_{\bar{m}m} \circ f_{\bar{m}}^{-1} \circ f_{m \nu} \circ f_\nu^{-1} = f_{\bar{m}m} \circ f_\nu^{-1},
\]
\[
g_{\bar{m}m} \circ f_{\bar{m}}^{-1} \circ f_{m \nu} \circ f_\nu^{-1} = -g_{\bar{m}m} \circ f_\nu^{-1},
\]
which hold on $[-1, 1]$ whenever
\[
\bar{\nu} = m \nu.
\]
(The $\pm$ sign in (3.7) is $+ at x$ if $g_{m \nu}(f_\nu^{-1}(x))^{\bar{m}}$ is positive and $- at x$ if $g_{m \nu}(f_\nu^{-1}(x))^{\bar{m}}$ is negative, owing to the branch cut structure of $\text{sn}^{-1}$.)
To flesh out the details, note that (3.8) holds for \( \nu = 1/\mu(\ell) \) and \( \tilde{\nu} = 1/\mu(\ell) \) if and only if
\[
\frac{K(\ell)}{K(\ell')} = \frac{K(\ell)}{mK(\ell')}.
\]
Comparing with (2.6), we see that this happens precisely when \( \ell = \lambda = F_m(\ell; \ell) \). In turn, this holds if and only if \( \ell = \cos \Theta \) and \( \tilde{\ell} = \cos \tilde{\Theta} \) with \( \tilde{\Theta} = |\arg(s_m(e^{i\Theta}; \Theta))| \).

Let us now compute \( \tilde{s}_m(s_m(z; \Theta); \tilde{\Theta}) \) under the assumption that \( \tilde{\Theta} = |\arg(s_m(e^{i\Theta}; \Theta))| \).

Since
\[
s_m(z; \Theta) = \tilde{F}_m(z; \Theta) + i\tilde{G}_m(z; \Theta) = f_{\nu m}(f_{\nu}^{-1}(x)) + i(\text{sign Im } z)^m g_{\nu m}(f_{\nu}^{-1}(x))
\]
and \( s_m(z; \Theta)^{-1} = \tilde{F}_m(z; \Theta) - i\tilde{G}_m(z; \Theta) \), we have
\[
\frac{1}{2}(s_m(z; \Theta) + s_m(z; \Theta)^{-1}) = \tilde{F}_m(z; \Theta) = f_{\nu m}(f_{\nu}^{-1}(x)),
\]
where
\[
x = \frac{1}{2}(z + z^{-1}), \quad \nu = \frac{1}{\mu(\ell)}, \quad \ell = \cos \Theta.
\]
Thus, denoting
\[
\tilde{\nu} = 1/\mu(\cos \tilde{\Theta}) = m\nu,
\]
\[
\sigma = (\text{sign Im } s_m(z; \Theta))^m = (\text{sign Im } z)^m \text{sign}(g_{\nu m}(f_{\nu}^{-1}(x)))^m,
\]
\[
\tau = \text{sign}(g_{\nu m}(f_{\nu}^{-1}(x)))^m,
\]
we find
\[
s_{\tilde{m}}(s_m(z; \Theta); \tilde{\Theta}) = f_{\tilde{\nu} m}(f_{\nu}^{-1}(f_{\nu m}(f_{\nu}^{-1}(x)))) + i\sigma g_{\tilde{m} m}(f_{\nu}^{-1}(f_{\nu m}(f_{\nu}^{-1}(x))))
\]
\[
= f_{\tilde{m} m}(f_{\nu}^{-1}(x)) + i\sigma g_{\tilde{m} m}(f_{\nu}^{-1}(x))
\]
\[
= s_{\tilde{m}}(z; \Theta),
\]
where the last line follows from the fact that $\sigma \tau = (\text{sign } \text{Im } z)^{\tilde{m}m}$. □

We illustrate Theorem 3.3 in Figure 4.

![Figure 4](image)

**Figure 4.** Illustration of Theorem 3.3 for $m = \tilde{m} = 3$, $\Theta = \pi/2 - 0.01$: $s_3(z; \Theta)$, $s_3(z; \tilde{\Theta})$ and $s_3(s_3(z; \Theta); \tilde{\Theta}) = s_9(z; \Theta)$. Only $[-\Theta, \Theta]$ is shown; by symmetry the plots look the same on $[\pi - \Theta, \pi + \Theta]$. Composing low-degree solutions results in a high-degree solution.

**Remark 3.4.** Theorem 3.3 can also be proved by counting equioscillation points. As $\theta$ runs from $-\Theta$ to $\Theta$, the number $\tilde{\theta} := \arg \left( s_m(e^{i\theta}; \Theta) \right)$ equioscillates $m + 1$ times, taking values in $[-\tilde{\Theta}, \tilde{\Theta}]$ and achieving its extrema at the endpoints. Each time $\tilde{\theta}$ runs from $\pm \tilde{\Theta}$ to $\mp \tilde{\Theta}$, the number

$$\hat{\theta} := \arg \left( s_{\tilde{m}}(e^{i\Theta}; \tilde{\Theta}) \right) = \arg \left( s_{\tilde{m}}(s_m(e^{i\theta}; \Theta); \tilde{\Theta}) \right)$$

equioscillates $\tilde{m} + 1$ times, achieving its extrema at the endpoints. By counting extrema, we see that as $\theta$ runs from $-\Theta$ to $\Theta$, $\hat{\theta}$ equioscillates $\tilde{m}m + 1$ times. Since $s_{\tilde{m}}(s_m(z; \Theta))$ is a rational function of degree $\tilde{m}m$, we can argue as we did in the proof of Theorem 2.1 that $s_{\tilde{m}}(s_m(z; \Theta))$ must be a solution of Problem Z6. Hence, $s_{\tilde{m}}(s_m(z; \Theta); \tilde{\Theta}) = s_{\tilde{m}m}(z; \Theta)^{\sigma}$ for some $\sigma \in \{-1, 1\}$. Evaluating both sides of this equation at $z = i$ yields $\sigma = 1$, so $s_{\tilde{m}}(s_m(z; \Theta); \tilde{\Theta}) = s_{\tilde{m}m}(z; \Theta)$.

**Remark 3.5.** The identity (3.6) shows that Zolotarev’s (scaled) minimax approximant $F_m(x; \ell)$ of $\text{sign}(x)$ on $[-1, -\ell] \cup [\ell, 1]$ satisfies

$$F_{\tilde{m}}(F_m(x; \ell); \tilde{\ell}) = F_{\tilde{m}m}(x; \ell)$$

whenever $\tilde{\ell} = F_m(\ell; \ell)$. This composition law has been studied in, for example, [3, 4, 16].
Remark 3.6. It is not hard to check that the function \( \tilde{s}_{2n+1}(z; \Theta) := s_{2n+1}(z; \Theta)(-1)^n \) also behaves nicely under composition: If \( \Theta = |\arg(s_{2n+1}(e^{i\Theta}); \Theta)| \), then

\[
\tilde{s}_{2n+1}(\tilde{s}_{2n+1}(z; \Theta); \Theta) = \tilde{s}_{2(2n+1)}(z; \Theta).
\]

Since

\[
\tilde{s}_{2n+1}(z; \Theta) = \frac{z}{r_n(z^2; \Theta)},
\]

we obtain from Theorem 3.3 an analogous composition law for solutions of Problem Z5.

Corollary 3.7. Let \( \Theta \in (0, \pi/2), \tilde{n}, n \in \mathbb{N}_0, \) and \( \tilde{\Theta} = |\arg(s_{2n+1}(e^{i\Theta}); \Theta)| = |\arg(e^{i\Theta}/r_n(e^{2i\Theta}; \Theta))|. \) Then

(3.10) \[
r_n(z; \Theta)\tilde{r}_n \left( \frac{z}{r_n(z; \Theta)} ; \tilde{\Theta} \right) = r_{2\tilde{n}+n}(z; \Theta).
\]

Remark 3.8. This behavior closely parallels the behavior of rational minimax approximants of \( \sqrt{x} \) on positive real intervals; see [6,7].

Proof. We have

\[
\frac{\sqrt{z}}{r_{2\tilde{n}+n+n}(z; \Theta)} = \tilde{s}_{4\tilde{n}+2\tilde{n}+2n+1}(\sqrt{z}; \Theta)
= \tilde{s}_{2n+1}(\tilde{s}_{2n+1}(\sqrt{z}; \Theta); \tilde{\Theta})
= \tilde{s}_{2n+1} \left( \frac{\sqrt{z}}{r_n(z; \Theta)} ; \tilde{\Theta} \right)
= \frac{\sqrt{z}}{r_n(z; \Theta)} \tilde{s}_n \left( \frac{z}{r_n(z; \Theta)} ; \tilde{\Theta} \right).
\]

Rearranging this yields (3.10). \( \square \)

3.3. Connections with other functions. We conclude this section by drawing a few connections between the solutions to Problems Z5-Z6 and other well-studied functions.

Finite Blaschke products. Ng and Tsang [17,18] study a finite Blaschke product that behaves nicely under composition and solves the extremal problem (3.2) for \( Z_m(E, F) \) with \( E = [-\sqrt{\ell}, \sqrt{\ell}] \) and \( F = (-\infty, -\frac{1}{\sqrt{\ell}}] \cup [\frac{1}{\sqrt{\ell}}, \infty) \). The function is

\[
h_m(z; \ell) = \prod_{j=1}^{m} \frac{z - c_j}{1 - c_j z},
\]

where

\[
c_j = \sqrt{\ell} \text{cn} \left( \frac{2j-1}{m} K(\ell), \ell \right) \frac{\text{dn} \left( \frac{2j-1}{m} K(\ell), \ell \right)}{n \left( \frac{2j-1}{m} K(\ell), \ell \right)}.
\]

They show that if \( \tilde{\ell} = Z_m([-\sqrt{\ell}, \sqrt{\ell}], (-\infty, -\frac{1}{\sqrt{\ell}}] \cup [\frac{1}{\sqrt{\ell}}, \infty)) \), then [17, Proposition 2]

\[
h_{\tilde{m}}(h_m(z; \ell); \tilde{\ell}) = h_{\tilde{m}m}(z; \ell)
\]

for any positive integers \( \tilde{m} \) and \( m \), and [18, Proposition 4.1(b)]

\[
\left( \frac{1 - \tilde{\ell}}{1 + \tilde{\ell}} \right) \frac{h_m(z; \ell) - 1}{h_m(z; \ell) + 1} = \frac{2}{1 + F_m(\kappa; \kappa)} F_m(x; \kappa),
\]
where
\[ x = \kappa \left( \frac{1 + \sqrt{\ell}}{1 - \sqrt{\ell}} \right) \frac{z - 1}{z + 1}, \quad \kappa = \left( \frac{1 - \sqrt{\ell}}{1 + \sqrt{\ell}} \right)^2. \]

Our function \( s_m \) is thus related to theirs via

\[
\frac{1 - \ell}{1 + \ell} \frac{h_m(z; \ell)}{h_m(z; \ell) + 1} = \frac{1}{1 + F_m(\kappa; \ell)} (s_m(w; \Phi) + s_m(w; \Phi)^{-1}),
\]

where
\[
\frac{1}{2} (w + w^{-1}) = \frac{1 - \sqrt{\ell}}{1 + \sqrt{\ell}} \frac{z - 1}{z + 1}, \quad \cos \Phi = \left( \frac{1 - \sqrt{\ell}}{1 + \sqrt{\ell}} \right)^2 = \kappa.
\]

**Padé approximants.** In the limit as \( \Theta \downarrow 0 \), the solution to Problem Z5 reduces to a Padé approximant of \( \sqrt{z} \). More precisely, let \( p_n(z) \) denote the type \((n, n)\) Padé approximant to \( \sqrt{z} \) at \( z = 1 \). An explicit formula for \( p_n(z) \) is [6, p. 707]

\[
p_n(z) = \sqrt{z} \frac{(1 + \sqrt{z})^{2n+1} + (1 - \sqrt{z})^{2n+1}}{(1 + \sqrt{z})^{2n+1} - (1 - \sqrt{z})^{2n+1}}.
\]

We say that a parametrized family of rational functions \( r(z; \Theta) \) converges coefficientwise to \( p_n(z) \) as \( \Theta \downarrow 0 \) if the coefficients in the numerator and denominator of \( r(z; \Theta) \), appropriately normalized, converge to those of \( p_n(z) \) as \( \Theta \downarrow 0 \).

**Proposition 3.9.** Let \( n \in \mathbb{N}_0 \). As \( \Theta \downarrow 0 \), \( r_n(z; \Theta) \) converges coefficientwise to \( p_n(z) \).

**Proof.** Since \( \left| r_n(z; \Theta) \right| = \left| p_n(z) \right| = 1 \) for all \( z \) with \( |z| = 1 \), it suffices to show that the poles of \( r_n(z; \Theta) \) approach the poles of \( p_n(z) \) as \( \Theta \downarrow 0 \). It is easy to check that the poles of \( p_n(z) \) are \( \left\{-\tan^2 \left( \frac{j\pi}{2n+1} \right) \right\}_{j=1}^n \). On the other hand, the poles of \( r_n(z; \Theta) \) are \( \{-a_j\}_{j=1}^n \). Since \( \lim_{\Theta \downarrow 0} K(\Theta) = K(0) = \pi/2 \), \( \lim_{\Theta \downarrow 0} \csc(z, \ell') = \csc(z, 0) = \csc z \), \( \lim_{\Theta \downarrow 0} \cot(z, \ell') = \cot(z, 0) = \cos z \), and \( \lim_{\Theta \downarrow 0} \cot(z, \ell') = \cot(z, 0) = 1 \) [5, Table 22.5.3], we have

\[
\lim_{\Theta \downarrow 0} a_j = \left( \frac{\sin(2j - 1)\omega + 1}{\cos(2j - 1)\omega} \right) 2^{-(1)^{j+n}},
\]

where \( \omega = \pi/(4n + 2) \). Using the identities \( \frac{\sin \theta + 1}{\cos \theta} = \cot \left( \frac{\pi}{4} - \theta \right) \) and \( \cot \left( \frac{\pi}{2} - \theta \right) = \tan \theta \), this can be simplified to

\[
\lim_{\Theta \downarrow 0} a_j = (\cot(n - j + 1)\omega)^{2^{-(1)^{j+n}}}
\]

\[
= \begin{cases} 
\tan^2(n - j + 1)\omega, & \text{if } j + n \text{ is odd}, \\
\tan^2(n + j)\omega, & \text{if } j + n \text{ is even}.
\end{cases}
\]

This shows that \( \{\lim_{\Theta \downarrow 0} a_j\}_{j=1}^n \) contains the squared tangent of every even multiple of \( \omega \). Hence,

\[
\{-\lim_{\Theta \downarrow 0} a_j\}_{j=1}^n = \left\{-\tan^2 \left( \frac{j\pi}{2n+1} \right) \right\}_{j=1}^n.
\]

\( \square \)
Chebyshev polynomials. It is interesting to note the similarity between the results in this paper and the defining property of the Chebyshev polynomials of the first kind $T_n(x)$:

$$Re(z^n) = T_n(Re z), \quad \text{if } |z| = 1.$$ 

In fact, we can write Theorem 2.4 in a more suggestive way by denoting

$$\mathcal{F}_m : [-1, 1] \times (0, 1) \to [-1, 1] \times (0, 1),$$

$$(x, \ell) \mapsto (F_m(x; \ell), F_m(\ell; \ell)),$$

$$\mathcal{S}_m : S \times S_+ \to S \times S_+,$$

$$(z, \zeta) \mapsto (s_m(z; |arg \zeta|), s_m(\zeta; |arg \zeta|)),$$

and

$$J : S \times S_+ \to [-1, 1] \times (0, 1),$$

$$(z, \zeta) \mapsto (Re z, Re \zeta),$$

where $S = \{z \in \mathbb{C} \mid |z| = 1\}$ and $S_+ = \{z \in S \mid 0 < Re z < 1\}$. With this notation, Theorem 2.4 says that

$$(3.11) \quad \mathcal{F}_m \circ J = J \circ \mathcal{S}_m,$$

and Theorem 3.3 says that

$$(3.12) \quad \mathcal{S}_{\tilde{m}} \circ \mathcal{S}_m = \mathcal{S}_{\tilde{m}m}.$$ 

By combining (3.11) with (3.12), we deduce

$$(3.13) \quad \mathcal{F}_{\tilde{m}} \circ \mathcal{F}_m = \mathcal{F}_{\tilde{m}m},$$

which is a restatement of (3.9). The identities (3.11-3.13) mimic the following identities involving the monomials $t_n(z) = z^n$ and the Chebyshev polynomials $T_n(x)$:

$$T_n \circ Re|_S = Re \circ t_n|_S, \quad t_m \circ t_n = t_{mn}, \quad T_m \circ T_n = T_{mn}.$$ 

References


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