# BEREZIN QUANTIZATION AND $K$-HOMOLOGY

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Abstract. The $E$-theory defined by Connes and Higson provides a realization of $K$-homology, the generalized homology theory dual to $K$-theory, based on the notion of asymptotic homomorphisms. With this realization it becomes possible to associate a $K$-homology element to a quantization scheme. In this article we associate an asymptotic homomorphism and $K$-homology element to the Berezin quantization of a bounded symmetric domain. Further, we identify this element with the element of $K$-homology defined by the Dolbeault operator of the domain.

## 1. Introduction

The $E$-theory defined by Connes and Higson [CH90] provides a realization of $K$-homology, the generalized homology theory dual to $K$-theory, based on the notion of asymptotic homomorphisms between $C^*$-algebras. The theory has found a number of applications to index theory through the work of Higson [Hig93] and Higson-Kasparov-Trout [HKT98]. Further, it plays an important role in the recent proof given by Higson-Kasparov of the Baum-Connes Conjecture for amenable groups [HK97, HK01].

The $E$-theory groups are defined to be certain groups of homotopy classes of asymptotic homomorphisms. In this paper we study the possibility of associating asymptotic homomorphisms, and hence elements of $E$-theory groups to quantization schemes. In this way $E$-theory groups become the receptacle of topological invariants of quantization schemes.

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In an earlier paper we initiated the study of the relationship between $E$-theory and quantization by analyzing the Berezin-Wick quantization of the complex plane from this perspective [Gue00]. The results of that work are rather satisfying. To the quantization scheme we associate an asymptotic homomorphism and thereby an element of an $E$-theory group. We further identify this element with the $E$-theory element defined by the $\overline{\partial}$-operator on the complex plane.

In this paper we continue this investigation by studying the Berezin quantization of bounded symmetric domains. We obtain results analogous to those described above for the Berezin-Wick quantization; the quantization defines an element of an appropriate $E$-theory group, and this element is identified with the element defined by the Dolbeault operator of the domain. Our main theorems are:

**Theorem A.** *Let $\Omega$ be a bounded symmetric domain. The Berezin–Toeplitz quantization defines an element of the $E$-homology of $\Omega$:*

$$[\,\mathit{Berezin}\,] \in E(C_0(\Omega), \mathbb{C}). \quad \Box$$

**Theorem B.** *Let $\Omega$ be a bounded symmetric domain. The $E$-homology class of the Berezin quantization equals the $E$-homology class of the Dolbeault operator:*

$$[\,\mathit{Berezin}\,] = [\,\mathit{Dolbeault}\,] \in E(C_0(\Omega), \mathbb{C}). \quad \Box$$

We remark that these classes are nonzero; indeed the $E$-theory group $E(C_0(\Omega), \mathbb{C})$ is isomorphic to the integers, and is generated by the class of the Dolbeault operator.

The main property of the Berezin quantization that is used in defining the $E$-homology element is that the Toeplitz operators used in its definition commute asymptotically as the value of Planck's constant tends to zero. This property was proven for a slightly restricted class of symbols in a paper of Borthwick-Lesniewski-Upmeier [BLU93, Thm. 2.2] through purely analytic means, and relying on the description of bounded symmetric domains in terms of Jordan algebras. The case of the Poincaré disk had been considered earlier by Klimek-Lesniewski [KL92, Theorem VI.2]. In the course of defining the $E$-homology element of the Berezin quantization we prove the following variant of these results using differential geometric techniques (we refer to Section 5 for definitions):

**Theorem C.** *Let $\Omega$ be a bounded symmetric domain. Let $\varphi$ and $\psi$ be continuous bounded functions on $\Omega$ and assume that $\varphi$ admits a continuous extensions to $\overline{\Omega}$. Then*

$$T_\hbar(\varphi\psi) - T_\hbar(\varphi)T_\hbar(\psi) \to 0, \quad \text{as } \hbar \to 0. \quad \square$$

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## 2. Bounded Symmetric Domains

We do not assume much familiarity with bounded symmetric domains, and include a brief summary of the relevant aspects of the theory. A more thorough introduction can be found in the books of Helgason and Krantz [Hel78, Kra92]; for more detailed information we refer to the books of Pijatetski-Shapiro, Hua, Loos and Mok [PS69, Hua63, Loo77, Mok89].

A domain $\Omega$ is an open connected subset of $\mathbb{C}^n$. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. Let $L^2(\Omega)$ be the space of measurable functions, square integrable with respect to Lebesgue measure. The Bergman space is denoted $H^2(\Omega)$ and is the subspace of $L^2(\Omega)$ consisting of the holomorphic functions. Since the domain is bounded $H^2(\Omega)$ contains the polynomials in the variables $z_1, \dots, z_n$ (holomorphic polynomials). Thus, $H^2(\Omega)$ is an infinite dimensional subspace of $L^2(\Omega)$. Further, it is closed.

The Bergman kernel function of $\Omega$ is defined by the formula

$$K(z, \overline{w}) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)}, \quad \{\varphi_n\} \text{ an orthonormal basis of } H^2(\Omega)$$

The Bergman kernel function is independent of the choice of orthonormal basis used to define it, and is holomorphic in $z$ and $\overline{w}$. Note that

$$K(z, \overline{z}) = \sum_i |\varphi_n(z)|^2, \quad \{\varphi_n\} \text{ an orthonormal basis of } H^2(\Omega) \tag{1}$$

is a real-valued function. Since we may take $\varphi_1(z) = 1/\sqrt{|\Omega|}$ we conclude that $K(z,\overline{z})$ is bounded below by $|\Omega|^{-1}$.

The Bergman kernel function is used to define the *infinitesimal Bergman metric*, the Hermitian structure on $\Omega$ defined by the bilinear form

$$h(z) = \sum_{ij} h_{ij}\, dz_i \otimes d\overline{z}_j, \quad h_{ij}(z) = \frac{\partial^2 \log K(z,\overline{z})}{\partial z_i \partial \overline{z}_j}, \tag{2}$$

(It follows from (1) that the form is Hermitian and positive semi-definite; the proof that it is positive definite is more complicated [Kra92], [Hel78, Ch. VIII, Prop. 3.4].) The associated $(1,1)$-form of the infinitesimal Bergman metric is

$$\omega = \frac{\sqrt{-1}}{2}\,\partial\overline{\partial}\,\log K(z,\overline{z}) = \frac{\sqrt{-1}}{2}\sum_{ij} h_{ij}\, dz_i \wedge d\overline{z}_j. \tag{3}$$

A Hermitian structure is *Kähler* if its associated $(1,1)$-form is closed. Thus, with its Hermitian structure $\Omega$ is a Kähler manifold. When applied to a bounded domain $\Omega$ differential geometric terms (isometry, completeness, etc) will always be interpreted with respect to the Riemannian structure underlying this Kähler structure.

**Proposition 2.1.** ([Hel78, Ch. VIII, Prop. 3.5]) Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. Holomorphic automorphisms of $\Omega$ are isometries.

A bounded domain $\Omega$ in $\mathbb{C}^n$ is *symmetric* if each point of the domain is the isolated fixed point of an involutive holomorphic automorphism of the domain; it is *homogeneous* if the group of its holomorphic automorphisms acts transitively. Every bounded symmetric domain is homogeneous [Loo77, Mok89].

Finally, observe that when equipped with the infinitesimal Bergman metric a bounded symmetric domain is a Hermitian symmetric space and, in particular, a complete manifold. Indeed, by the previous proposition, holomorphic automorphisms are isometries.

We require two further facts concerning the geometry of bounded symmetric domains. The first is that the volume form of a bounded symmetric domain is, up to a constant, the product of the Bergman kernel function and Lebesgue meausure. We can safely ignore the constant when considering only one domain at a time. Note that the following lemma remains valid in the broader setting of homogeneous domains.

**Lemma 2.2.** ([Hel78, Ch. VIII, Prop. 2.5 & 3.6], [Ber74, Thm. 5.1]) Let $\Omega$ be a bounded symmetric domain and let $d\mathrm{vol}$ denote its volume form computed with respect to the induced Riemannian structure. Then

$$d\mathrm{vol} = k_\Omega \, K(z,\overline{z}) \, dx_1 \wedge dy_1 \wedge \ldots dx_n \wedge dy_n,$$

where $k_\Omega$ is a constant depending only on the domain $\Omega$. □

The second and final fact we require is the following result of Donnelly (compare [Don97, Prop. 3.2] which is actually more general than the result stated here). By combining it with an elegant method of Gromov [Gro91], Donnelly proved a vanishing theorem for the Dolbeault operator. We will use the result to provide estimates on spectral functions of twisted Dolbeault operators. We thank M. Ramachandran for bringing this result to our attention; since it is not completely explicit in Donnelly's paper we include a sketch of the proof.

**Theorem 2.3.** *Let $\Omega$ be a bounded symmetric domain. The 1-form $\overline{\partial} \log K(z,\overline{z})$ is bounded (pointwise uniformly with respect to the metric on the cotangent space induced by the Kähler metric on $\Omega$).*

*Sketch of Proof.* Let $\alpha(z) = \overline{\partial} \log K(z,\overline{z})$. The idea of the proof is to use the transitive group of holomorphic automorphisms of $\Omega$ and the transformation law for the Bergman kernel with respect to such automorphisms [Hel78, Ch. VIII.3] to equate the norm of $\alpha(z)$ to the norm of $\alpha(0)$ plus a correction term. Precisely, if $\varphi_z$ is a holomorphic automorphism of $\Omega$ mapping the origin to $z$ then

$$K(u,\overline{w}) = K(\varphi_z(u), \overline{\varphi_z(w)}) J_{\varphi_z}(u) \overline{J_{\varphi_z}(w)}, \quad \text{for all } u, w \in \Omega,$$

where $J_{\varphi_z}$ is the complex Jacobian of $\varphi_z$. In particular,

$$\begin{aligned} \|\alpha(z)\| = \|(\varphi_z^* \alpha)(0)\| &= \|\overline{\partial} \log K(\varphi_z(w), \overline{\varphi_z(w)})(0)\| \\ &\leq 2\|\overline{\partial} \log J_{\varphi_z}(0)\| + \|\alpha(0)\|. \end{aligned}$$

Up to this point we have used only that the domain is homogeneous; it remains to show that for a bounded symmetric domain $\|\overline{\partial}\log J_{\varphi_z}(0)\|$ is bounded independently of $z \in \Omega$. This follows from more general facts as described by Donnelly [Don97]. Alternatively, when dealing with a specific domain, direct and elementary verification of this fact is often possible using precise knowledge of $\varphi_z$. $\square$

## 3. Berezin Quantization

In this section we review the construction of the Berezin quantization of bounded symmetric domains. The primary sources for this material are the original papers of Berezin [Ber74, Ber75a, Ber75b].

Let $\Omega$ be a bounded symmetric domain and let $K(z,\overline{z})$ be the Bergman kernel of $\Omega$ defined in the previous section. Define a family of measures on $\Omega$ by

$$d\mu_\hbar = c(\hbar)\, K(z,\overline{z})^{1-1/\hbar}\, d\lambda,$$

where $d\lambda$ is the ordinary Lebesgue measure and $c(\hbar)$ is a normalization constant that insures that the $\mu_\hbar$-measure of $\Omega$ is one. It is worth noting that for $\hbar = 1$ we obtain the normalized Lebesgue measure on $\Omega$; as $\hbar \to 0$ the measure concentrates at the origin.

For each of the measures $d\mu_\hbar$ we consider the space of measurable square integrable functions, as well as the subspace of holomorphic (square-integrable) functions:

$$\begin{aligned} L^2_\hbar(\Omega) &= \{\,\text{functions square-integrable with respect to } d\mu_\hbar\,\} \\ H^2_\hbar(\Omega) &= \{\,\text{holomorphic functions in } L^2_\hbar(\Omega)\,\}. \end{aligned}$$

Let $C_0(\Omega)$ denote the $C^*$-algebra of continuous functions on $\Omega$ vanishing at infinity. For $\varphi \in C_0(\Omega)$ we define the *Toeplitz operator* on $H^2_\hbar(\Omega)$ with *symbol* $\varphi$ to be the composition

$$H^2_\hbar(\Omega) \xrightarrow{\text{multiply by } \varphi} L^2_\hbar(\Omega) \xrightarrow{\text{project}} H^2_\hbar(\Omega). \tag{4}$$

Denoting the Hilbert space projection from $L^2_\hbar(\Omega)$ to $H^2_\hbar(\Omega)$ by $Q_\hbar$ we have $T_\hbar(\varphi)(u) = Q_\hbar(\varphi u)$, for all $u \in H^2_\hbar(\Omega)$. The Berezin quantization is defined by associating to the function $\varphi \in C_0(\Omega)$ the family of Toeplitz operators on $H^2_\hbar(\Omega)$:

$$\mathcal{B}(\varphi) = \{\, T_\hbar(\varphi) \,\}. \tag{5}$$

(This definition is not quite the same as the one given by Berezin [Ber74, Ber75a]; instead of considering Toeplitz operators themselves he associates to them their *contravariant symbols*, functions on $\Omega$. In this way he constructs a family of products on $C_c^\infty(\Omega)$ which form his quantization.)

Our first goal is to define the $E$-homology class of the Berezin quantization by associating to it a generalized asymptotic morphism. The content of this statement is:

(i) $T_\hbar : C_0(\Omega) \to \mathcal{K}(H^2_\hbar(\Omega))$ is a $*$-linear contraction,
(ii) $T_\hbar(\varphi)T_\hbar(\psi) - T_\hbar(\varphi\psi) \to 0$ as $h \to 0$ (in norm in the respective $\mathcal{B}(H^2_\hbar(\Omega))$),

and further that we can endow the collection of Hilbert spaces $H^2_\hbar(\Omega)$ with the structure of a continuous field of Hilbert spaces $\{\, H^2_\hbar(\Omega) \,\}$ in such a way that

(iii) the family of Toeplitz operators $\{\, T_\hbar(\varphi) \,\}$ is a continuous section of the field of elementary $C^*$-algebras associated to the field $\{\, H^2_\hbar(\Omega) \,\}$.

Note that Property (i) is a standard consequence of the Toeplitz construction. Each of properties (ii) and (iii) will be established in Section 5 (see Thms. 5.7 and 5.8); the discussion will be based on spectral properties of a family of twisted Dolbeault operators on $\Omega$.

We pause to remark that, although we will not do so, it is possible to give analytic proofs of properties (ii) and (iii) as well. Indeed, property (ii) is a weak form of the correspondence principle proved by Borthwick-Lesniewski-Upmeier [BLU93, Thm. 2.2].

## 4. Vanishing Theorems

Let $\Omega$ be a bounded symmetric domain in $\mathbb{C}^n$. As usual, consider $\Omega$ equipped with the infinitesimal Bergman metric (2) with respect to which it is a complete Kähler manifold.

Define a family of Hermitian holomorphic line bundles $E_\hbar$ on $\Omega$. As holomorphic line bundles these are all the trivial bundle. A Hermitian structure on a such a line bundle is defined by a real-valued function giving the square of the length of $1 \in \mathbb{C}$ at the point $z \in \Omega$. The bundles $E_\hbar$ differ in their Hermitian structures, which are defined by

$$|1|_\hbar^2(z) = k_\Omega^{-1} c(\hbar) \cdot K(z, \overline{z})^{-1/h}.$$

Equip the $E_\hbar$ with the unique connexions compatible with their complex and Hermitian structures. Denote the curvature form of $E_\hbar$ with this connexion by $\Theta_\hbar$.

**Lemma 4.1.** *The curvature forms $\Theta_\hbar$ of the bundles $E_\hbar$ satisfy*

$$\frac{\sqrt{-1}}{2}\,\Theta_\hbar = \frac{1}{\hbar}\,\omega. \tag{6}$$

*Proof.* This follows straightforwardly from the definitions. If $s$ is the section of $E_\hbar$ whose value at $z \in \Omega$ is $1 \in \mathbb{C}$ then both

(i) $|s(z)|^2 = \text{constant} \cdot K(z, \overline{z})^{-1/\hbar}$, and
(ii) $\Theta_\hbar = \overline{\partial}\partial \log |s(z)|^2$.

Comparing with (3) we obtain the result. $\square$

We are interested in the Dolbeault operator of $\Omega$ twisted by the line bundles $E_\hbar$ and recall the definition of these operators. Denote by

$$\begin{aligned} A_c^{pq} &= \{\, \text{compactly supported } (p,q)\text{-forms} \,\} \\ A_c^{pq}(E_\hbar) &= \{\, \text{compactly supported } E_\hbar\text{-valued } (p,q)\text{-forms} \,\} \end{aligned}$$

(these spaces differ only in their pre-Hilbert inner products). The $\overline{\partial}$-operator has a canonical extension to forms with values in any holomorphic vector bundle. In particular, we have for all $p = 0, \ldots, n$

$$\overline{\partial} : \bigoplus_q A_c^{pq}(E_\hbar) \to \bigoplus_q A_c^{pq}(E_\hbar)$$

and $\overline{\partial}$ maps $A_c^{pq}(E_\hbar)$ to $A_c^{p,q+1}(E_\hbar)$. Each of the spaces $A_c^{pq}(E_\hbar)$ has an Hermitian inner product induced by the Hermitian structures of $\Omega$ and $E_\hbar$. The operator $\overline{\partial}$ has a formal adjoint which we shall denote by $\overline{\partial}^*_\hbar$. Notice that although $\overline{\partial}$ does not depend on the Hermitian structure $\overline{\partial}^*_\hbar$ does.

Finally, the Hilbert space completions of the spaces of compactly supported forms will be denoted

$$\begin{aligned} \mathcal{A}^{pq} &= \text{Hilbert space of } (p,q)\text{-forms} \\ \mathcal{A}^{pq}(E_\hbar) &= \text{Hilbert space of } E_\hbar\text{-valued } (p,q)\text{-forms.} \end{aligned}$$

The following two special instances of these notations are worth mentioning explicitly; it follows from Lemma 2.2 that

$$L^2_\hbar(\Omega) = \mathcal{A}^{00}(E_\hbar) \tag{7}$$

$$H^2_\hbar(\Omega) = \{\text{ holomorphic sections of } E_\hbar\,\} = \ker\left(\overline{\partial} : \mathcal{A}^{00}(E_\hbar) \to \mathcal{A}^{01}(E_\hbar)\right). \tag{8}$$

Since $\Omega$ is a complete manifold the formally self-adjoint operator $\overline{\partial}+\overline{\partial}^*_\hbar$ is essentially self-adjoint. The *twisted Dolbealt operators* $D_\hbar$ are the closures of the operators $\overline{\partial}+\overline{\partial}^*_\hbar$. They are self-adjoint unbounded operators (one for each $p = 0, \ldots, n$):

$$D_\hbar : \bigoplus_q \mathcal{A}^{pq}(E_\hbar) \to \bigoplus_q \mathcal{A}^{pq}(E_\hbar).$$

The *Dolbeault Laplacian* is $\Box_\hbar = D^2_\hbar$. It preserves the bidegree of forms and when restricted to $(p,q)$-forms is denoted $\Box^{pq}_\hbar$.

We are interested in vanishing theorems for the twisted Dolbeault operators $D_\hbar$. They have the consequence that the kernel of this operator is concentrated in degree zero, so that by (8) the quantization space $H^2_\hbar(\Omega)$ is in fact the kernel of $D_\hbar$ ($p = 0$). The vanishing theorem is obtained using the standard Bochner method; following Roe [Roe88] we employ an adaptation of the Bochner method to the current setting of complete manifolds. We give a short review for the purposes of which we suppress the subscript in all notations, writing simply $E$ for a Hermitian holomorphic line bundle, $\Theta$ for the curvature of its canonical connexion, etc. As is standard, denote by $L$ the operator of exterior product with $\omega$, the $(1,1)$-form associated to the Kähler structure of $\Omega$. Further, denote the unique connexion on $E$ compatible with its Hermitian and complex structure by $\nabla + \overline{\partial}$. Using square brackets to denote commutators the Kähler identities [GH78] (compare [GH78, Ch. 0.7, p. 111]) are

$$[L^*, \overline{\partial}] = -\frac{\sqrt{-1}}{2}\,\nabla^* \quad \text{and} \quad [L^*, \nabla] = \frac{\sqrt{-1}}{2}\,\overline{\partial}^*.$$

From these and the fact that $\Theta = \left(\nabla + \overline{\partial}\right)^2 = \nabla\overline{\partial} + \overline{\partial}\nabla$ it is a calculation to obtain the basic Bochner identity

$$\Box^{pq} = \nabla^*\nabla + \nabla\nabla^* - 2\sqrt{-1}[L^*, \Theta]. \tag{9}$$

As a final bit of notation denote the space of $\overline{\partial}$-harmonic $(p, q)$-forms with values in $E$ by $\mathcal{H}^{pq}(E)$;

$$\mathcal{H}^{pq}(E) = \text{kernel of } \Box^{pq}_E.$$

**Theorem 4.2** (First Vanishing Theorem). *Let $E$ be a Hermitian holomorphic line bundle on $\Omega$. If the curvature $\Theta$ of the canonical connection of $E$ satisfies*

$$\frac{\sqrt{-1}}{2}\,\Theta = \lambda\omega$$

*for some $\lambda > 0$ then the spectrum of the Dolbeault Laplacian on $(p, q)$-forms $\Box^{pq}$ is bounded below by $4\lambda(p + q - n)$. In particular,*

$$\mathcal{H}^{pq}(E) = 0 \quad \textit{if } p + q > n.$$

*Proof.* Combining our assumption with the identity [GH78, Ch. 0.7, p. 121]

$$[L^*, L] = (n - p - q), \quad \text{on } (p, q)\text{-forms}$$

and the basic Bochner identity (9) we obtain

$$\begin{aligned} \Box^{pq} &= \nabla^*\nabla + \nabla\nabla^* - 4\lambda[L^*, L] \\ &= \nabla^*\nabla + \nabla\nabla^* + 4\lambda(p + q - n), \quad \text{on } (p, q)\text{-forms.} \quad \Box \end{aligned}$$

**Theorem 4.3** (Second Vanishing Theorem). *Let $D_\hbar$ denote the Dolbeault operator $(p = 0)$ of $\Omega$ with values in the Hermitian holomorphic line bundle $E_\hbar$, $\hbar \in (0, 1)$. Then*

$$\mathcal{H}^{0q}(E_\hbar) = 0, \quad \textit{for } q > 0$$

*and on the orthogonal complement of its kernel $D_\hbar$ is bounded below by $2\sqrt{1/\hbar - 1}$.*

*Proof.* Use the canonical line bundle and its dual;

$$K_\Omega = \bigwedge^{n0}\Omega = \mathbb{C}\{\, dz_1 \wedge \cdots \wedge dz_n \,\}, \quad \text{and} \quad K^*_\Omega = \mathbb{C}\left\{ \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n} \right\}.$$

The metric on $K^*_\Omega$ induced from the Kähler metric on $\Omega$ is given by

$$\left| \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n} \right|^2_\hbar (z) = (\text{constant}) \cdot K(z, \overline{z}),$$

and its curvature form therefore satisfies $\frac{\sqrt{-1}}{2}\Theta_{K^*} = -\omega$. Hence the curvature form of $E_\hbar \otimes K^*$ satisfies

$$\frac{\sqrt{-1}}{2}\,\Theta_{E_\hbar \otimes K^*} = \frac{\sqrt{-1}}{2}\,(\Theta_\hbar + \Theta_{K^*}) = \left(\frac{1}{\hbar} - 1\right)\omega.$$

Apply the previous theorem to conclude (the second equality is just a definition)

$$\begin{aligned} \mathcal{H}^{0q}(E_\hbar) &= \mathcal{H}^{0q}(E_\hbar \otimes K^* \otimes K) \\ &= \mathcal{H}^{nq}(E_\hbar \otimes K^*) \\ &= 0, \quad \text{if } q > 0. \end{aligned}$$

The remainder of the proposition follows from Convergence Transfer ([Roe88] or Section 3 of [Gue98]); if $\Box_\hbar = D_\hbar^2$ is bounded below by $4(1/\hbar - 1)$ on $\bigoplus_{q=1,3,\ldots} \mathcal{A}^{pq}(E_\hbar)$ then $D_\hbar$ is bounded below by $2\sqrt{1/\hbar - 1}$ on the orthogonal complement of its kernel. $\Box$

**Corollary 4.4.** *The quantization spaces $H^2_\hbar(\Omega)$ for the Berezin quantization are the kernels of the twisted Dolbeault operators $D_\hbar$;*

$$H^2_\hbar(\Omega) = \ker D_\hbar = \mathcal{H}^{00}(E_\hbar)$$

*Proof.* This follows immediately from the previous theorem and (8). $\Box$

## 5. $E$-Theory Elements

This section is devoted to the construction of an $E$-theory element corresponding to the Berezin quantization. We will realize this $E$-theory element as the homotopy class of a generalized asymptotic morphism as defined in the appendix.

To obtain a generalized asymptotic morphism we provide the collection of Hilbert spaces $H^2_\hbar(\Omega)$, for $\hbar \in (0,1]$, with the structure of a continuous field of Hilbert spaces. We do this in a very explicit and concrete manner. Although the results are actually stronger than required for our immediate goal of defining and analyzing the $E$-theory element associated to the quantization they do allow an elementary analysis based on estimates of functions of the twisted Dolbeault operators; we believe them to be of independent interest.

We introduce several convenient pieces of notation. Denote by $E$ the trivial Hermitian holomorphic line bundle for which $|1|(z) = 1$ and by

$$\mathcal{A} = \bigoplus_q \mathcal{A}^{0q}, \quad \text{and} \quad \mathcal{A}_\hbar = \bigoplus_q \mathcal{A}^{0q}(E_\hbar).$$

the Hilbert spaces of $E$ and $E_\hbar$-valued differential forms. The maps $E_\hbar \to E$ of multiplication by the functions

$$u_\hbar = k_\Omega^{-1/2} c(\hbar)^{1/2} \cdot K(z,\overline{z})^{-1/2\hbar}$$

are unitary bundle isomorphisms and induce unitary isomorphisms $U_\hbar : \mathcal{A}_\hbar \to \mathcal{A}$ of the Hilbert spaces of forms which preserve the spaces of compactly supported forms. The $U_\hbar$ will be used to define and trivialize the field $\{\, H^2_\hbar(\Omega) \,\}$. We must, however, address the fact that the functions $u_\hbar$ are not holomorphic and the $U_\hbar$ do not preserve the subspaces of holomorphic sections, and in particular do not preserve the quantization spaces $H^2_\hbar(\Omega)$.

Recall that the *interior product* with a smooth differential form $\tau$ is the negative of the adjoint of the exterior product with $\tau$. We introduce the notation $\tau \lrcorner\, (\cdot)$ for the interior product with $\tau$.

**Proposition 5.1.** *Define $V : \mathcal{A} \to \mathcal{A}$ on the domain of smooth compactly supported forms by*

$$V\sigma = \big(\overline{\partial}\log K(z,\overline{z})\big) \wedge \sigma - \big(\overline{\partial}\log K(z,\overline{z})\big) \lrcorner\, \sigma, \quad \sigma \in A_c^{pq}.$$

*The following diagram, in which all unbounded operators are defined on the domain of smooth compactly supported forms,*

$$\begin{array}{ccc} \mathcal{A}_\hbar & \xrightarrow{U_\hbar} & \mathcal{A} \\ {\scriptstyle D_\hbar=\overline{\partial}+\overline{\partial}_\hbar^*}\downarrow & & \downarrow{\scriptstyle D+V/2\hbar} \\ \mathcal{A}_\hbar & \xrightarrow[U_\hbar]{} & \mathcal{A}. \end{array}$$

*commutes (the domains are preserved by $U_\hbar$).*

*Proof.* Calculate for a smooth compactly supported form $\sigma$;

$$\begin{aligned} U_\hbar\overline{\partial}U_\hbar^*(\sigma) &= U_\hbar\overline{\partial}(u_\hbar^{-1}\sigma) \\ &= U_\hbar(\overline{\partial}u_\hbar^{-1}\wedge\sigma + u_\hbar^{-1}\overline{\partial}\sigma) \\ &= u_\hbar\overline{\partial}u_\hbar^{-1}\wedge\sigma + \overline{\partial}\sigma \\ &= -u_\hbar^{-1}\overline{\partial}u_\hbar\wedge\sigma + \overline{\partial}\sigma \\ &= -(\overline{\partial}\log u_\hbar)\wedge\sigma + \overline{\partial}\sigma, \end{aligned}$$

from which follows

$$U_\hbar\overline{\partial}U_\hbar^* = -(\overline{\partial}\log u_\hbar)\wedge(\cdot) + \overline{\partial}. \tag{10}$$

Taking the adjoint in (10) we obtain

$$U_\hbar\overline{\partial}_\hbar^*U_\hbar^* = (\overline{\partial}\log u_\hbar)\lrcorner(\cdot) + \overline{\partial}^*, \tag{11}$$

where, of course, $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$ on the space $\mathcal{A}$.

Notice that

$$\overline{\partial}\log u_\hbar = \overline{\partial}\log k_\Omega^{-1/2}c(\hbar)^{1/2}K(z,\overline{z})^{-1/2h} = \frac{-1}{2\hbar}\,\overline{\partial}\log K(z,\overline{z})$$

so that for smooth compactly supported forms $\sigma$ we have

$$\frac{1}{2\hbar} V\sigma = -\overline{\partial} \log u_\hbar \wedge \sigma + \overline{\partial} \log u_\hbar \lrcorner\, \sigma, \quad \sigma \in A_c^{pq}(E). \tag{12}$$

Adding (10) and (11) and employing (12) we obtain

$$\begin{aligned} U_\hbar D_\hbar U_\hbar^* &= U_\hbar(\overline{\partial} + \overline{\partial}_\hbar^*)U_\hbar^* \\ &= -(\overline{\partial} \log u_\hbar) \wedge (\cdot) + (\overline{\partial} \log u_\hbar) \lrcorner (\cdot) + D \\ &= D + \frac{1}{2\hbar} V, \end{aligned}$$

where, of course, $D$ is the ordinary Dolbeault operator of $\Omega$ computed with respect to its Kähler metric. □

The domain $\bigoplus_q A_c^{0q}$ is a common core for the operators

$$U_\hbar D_\hbar U_\hbar^* = D + \frac{1}{2\hbar} V.$$

They extend to self-adjoint unbounded operators on (a common domain in) $\mathcal{A}$.

**Proposition 5.2.** *For $\hbar \in (0,1)$ the projections $P_\hbar$ onto the kernels of $D + \frac{1}{2\hbar}V$ form a norm continuous family of projections on the Hilbert space $\mathcal{A}$.*

*Proof.* The operators $D + \frac{1}{2\hbar} V$ have the same spectral properties as the $D_\hbar$ (see Theorem 4.3). Consequently, the projections $P_\hbar$ can be realized as continuous spectral functions of these operators, and in fact, they can be realized simultaneously (ie, using a single continuous function); given $\hbar_0 < 1$, since the operators $D + \frac{1}{2\hbar} V$ are bounded below on the orthogonal complements of their kernels independently of $\hbar \in (0, \hbar_0]$, there exists $f \in C_0(\mathbb{R})$ such that

$$P_\hbar = f(D + \frac{1}{2\hbar} V), \quad \text{for all } \hbar \in (0, \hbar_0]. \tag{13}$$

The proposition follows from this equality together with the following two lemmas (Lemma 5.4 allows us to apply Lemma 5.3). □

**Lemma 5.3.** *Let $T$ be a self-adjoint, unbounded operator and $A$ be a self-adjoint bounded operator on a Hilbert space. For all $f \in C_0(\mathbb{R})$ the operator-valued function*

$$t \mapsto f(T + tA) : (0, \infty) \to \mathcal{B}(H)$$

*is continuous in norm.*

*Proof.* The set

$$\{\, f \in C_0(\mathbb{R}) : f(T + tA) \text{ is continuous in } t \,\}$$

is a $C^*$-subalgebra of $C_0(\mathbb{R})$. The proof is concluded by showing that it is all of $C_0(\mathbb{R})$ which follows from showing that it contains the resolvent functions $r_\pm(x) = \left(x \pm \sqrt{-1}\right)^{-1}$. This is a simple calculation; the norm of

$$r_\pm(T + tA) - r_\pm(T + t'A) = r_\pm(T + tA)\,(t' - t)\,Ar_\pm(T + t'A)$$

is bounded by $|t - t'|\|A\|$. □

**Lemma 5.4.** *The potential $V$ is pointwise uniformly bounded on $\Omega$.*

We will reduce the lemma to Theorem 2.3 but must prepare for the proof by introducing some notation. For $\xi \in T^*_{\mathbb{R}}\Omega \otimes \mathbb{C}$ denote $c(\xi)$ and $\widetilde{c}(\xi)$ the sum and difference of exterior and interior multiplication with $\xi$, respectively;

$$\begin{aligned} c(\xi) &= \xi \wedge (\cdot) + \xi \lrcorner (\cdot) \\ \widetilde{c}(\xi) &= \xi \wedge (\cdot) - \xi \lrcorner (\cdot). \end{aligned}$$

Each of $c(\xi)$ and $\widetilde{c}(\xi)$ are complex linear endomorphisms of $\bigwedge^* T^*_{\mathbb{R}}\Omega \otimes \mathbb{C}$. Note that $\xi \wedge (\cdot)$ is complex-linear in $\xi$ whereas $\xi \lrcorner (\cdot)$ is complex-antilinear in $\xi$. From this simple observation it follows that

$$\begin{aligned} \widetilde{c}(\sqrt{-1}\xi)\sigma &= (\sqrt{-1}\xi) \wedge \sigma - (\sqrt{-1}\xi) \lrcorner \sigma \\ &= \sqrt{-1}\xi \wedge \sigma + \sqrt{-1}\xi \lrcorner \sigma \\ &= \sqrt{-1} c(\xi)\sigma \end{aligned}$$

so that $\widetilde{c}(\sqrt{-1}\xi) = \sqrt{-1}c(\xi)$ and $c(\sqrt{-1}\xi) = \sqrt{-1}\widetilde{c}(\xi)$. For $\xi$ and $\eta \in T^*_{\mathbb{R}}\Omega$ we have

$$c(\xi)^2 = -\|\xi\|^2 \tag{14}$$

$$\widetilde{c}(\xi)^2 = \|\xi\|^2 \tag{15}$$

$$c(\xi)\widetilde{c}(\eta) + \widetilde{c}(\eta)c(\xi) = 0. \tag{16}$$

Now $T^*_{\mathbb{R}}\Omega \otimes \mathbb{C} = T^*_{\mathbb{R}}\Omega \oplus \sqrt{-1}T^*_{\mathbb{R}}\Omega$ is an orthogonal sum (in the complexified metric coming from the Riemannian metric on $T^*_{\mathbb{R}}\Omega$ underlying its Kähler metric) and the inclusions of $T^*_{\mathbb{R}}\Omega$ into the first and second factors are isometric and real-linear.

Combining all the above facts we conclude that (14) holds also for complex cotangent vectors;

$$\begin{aligned} c(\alpha + \sqrt{-1}\beta)^2 &= \big(c(\alpha) + c(\sqrt{-1}\beta)\big)^2 \\ &= \big(c(\alpha) + \sqrt{-1}\widetilde{c}(\beta)\big)^2 \\ &= c(\alpha)^2 - \widetilde{c}(\beta)^2 + \sqrt{-1}\,(c(\alpha)\widetilde{c}(\beta) + \widetilde{c}(\beta)c(\alpha)) \\ &= -\|\alpha\|^2 - \|\beta\|^2 \\ &= -\|\alpha + \sqrt{-1}\beta\|^2. \end{aligned}$$

Similarly one shows that $\widetilde{c}(\alpha + \sqrt{-1}\beta)^2 = \|\alpha + \sqrt{-1}\beta\|^2$.

The consequence of this discussion that we require is that if $f$ is a real-valued smooth function on $\Omega$ then the endomorphism $\widetilde{c}(\overline{\partial} f)$ of $\oplus_n A^n_c$ satisfies

$$\widetilde{c}(\overline{\partial} f)^2 = \|\overline{\partial} f\|^2 = \frac{1}{2}\|df\|^2. \tag{17}$$

*Proof of Lemma 5.4.* We reduce the statement to Theorem 2.3. In the notation of the previous discussion $V = \widetilde{c}(\overline{\partial}\log K(z,\overline{z}))$. It follows from (17) that

$$V^2 = \|\overline{\partial}\log K(z,\overline{z})\|^2,$$

meaning that the square of the bundle endomorphism $V$ is multiplication by the function. Thus, the norm of $V$ is bounded by the supremum norm of $\overline{\partial}\log K(z,\overline{z})$ which by Theorem 2.3 is finite. □

We begin the procedure of associating a generalized asymptotic morphism to the Berezin quantization by endowing the collection of Hilbert spaces $H^2_\hbar(\Omega)$ with the structure of a trivial continuous field. We have defined a family a unitary isomorphisms $U_\hbar : \mathcal{A}_\hbar \to \mathcal{A}$ which we restrict to isometries $H^2_\hbar(\Omega) \to \mathcal{A}$. We have observed in Corollary 4.4 that

$$\ker(D_\hbar) = H^2_\hbar(\Omega) \subset \mathcal{A}_\hbar$$

are a family of closed linear subspaces, and further in Proposition 5.2 that there exists a norm continuous family of projections $\{ P_\hbar \}$ on $\mathcal{A}$ such that

$$U_\hbar(H^2_\hbar(\Omega)) = P_\hbar \mathcal{A}.$$

In other words, the range projections of the $U_\hbar$ are the $P_\hbar$ and these form a norm continuous family of projections on $\mathcal{A}$. Thus the hypothesis of the Lemma 7.1 are satisfied and we have proven the following

**Proposition 5.5.** *Let $\Gamma$ be the collection of functions $x$ of $\hbar \in (0,1)$ such that $x(\hbar) \in H^2_\hbar(\Omega)$ and $U_\hbar x(\hbar)$ is a continuous function of $\hbar$, where we view $U_\hbar$ as the isometry $H^2_\hbar(\Omega) \to P_\hbar \mathcal{A} \hookrightarrow \mathcal{A}$. Then $\Gamma$ defines the structure of a trivial field of Hilbert spaces $\{ H^2_\hbar(\Omega) \}$.* □

*Remark.* The continuous field defined in the proposition is generated (in the sense of [Dix70, 10.2.3]) by the collection of constant functions of $\hbar$ valued in the holomorphic polynomials. The triviality of the field follows from the general theory of continuous fields once we note that each of the spaces $H^2_\hbar(\Omega)$ is infinite dimensional [Dix70, 10.8.7].

For the remainder of this section denote by $\mathcal{K}_\hbar$ and $\mathcal{K}$ the $C^*$-algebras of compact operators on $\mathcal{A}_\hbar$ and $\mathcal{A}$, respectively. As an immediate consequence of Lemma 7.2 we obtain the following characterization of the continuous sections of the field of elementary $C^*$-algebras associated to the field $\{ H^2_\hbar(\Omega) \}$.

**Proposition 5.6.** *Employ the notation of Proposition 5.5. A function $K(\hbar)$ such that $K(\hbar) \in \mathcal{K}_\hbar$ is a continuous section of the field of elementary $C^*$-algebras $\{ K_\hbar \}$*

*associated to the field* $\{ H^2_\hbar(\Omega) \}$ *if and only if* $U_\hbar K(\hbar) U^*_\hbar$ *is a continuous function of* $\hbar$ *with values in* $\mathcal{K}$. □

We come to the main theorem of this section; we associate to the Berezin quantization a generalized asymptotic morphism. Recall that the Berezin quantization is defined by associating to $\varphi \in C_0(\Omega)$ the family of Toeplitz operators $\{ T_\hbar(\varphi) \}$ on the family of Hilbert spaces $\{ H^2_\hbar(\Omega) \}$. We denote this family by $\mathcal{B}(\varphi) = \{ T_\hbar(\varphi) \}$.

**Theorem 5.7.** *The Berezin quantization defines a generalized asymptotic morphism*

$$\varphi \mapsto \mathcal{B}(\varphi) = \{ T_\hbar(\varphi) \} : C_0(\Omega) \to C_b\{ \mathcal{K}(H^2_\hbar(\Omega)) \}.$$

*As described in the appendix, this generalized asymptotic morphism determines an element of the E-homology of* $\Omega$*;*

$$[\mathcal{B}] = \text{ homotopy class of } \widetilde{\varphi}_t \ \in E_0(\Omega).$$

*Proof.* Via the inclusion $H^2_\hbar(\Omega) = \ker(D_\hbar) \hookrightarrow \mathcal{A}_\hbar$ the Toeplitz operator $T_\hbar(\varphi)$ is viewed as the compression of the multiplication operator $M_\varphi$ to the subspace $H^2_\hbar(\Omega)$:

$$T_\hbar(\varphi) = Q_\hbar M_\varphi Q_\hbar,$$

where $Q_\hbar$ is the projection onto the kernel of $D_\hbar$.

As the compression of the $*$-homomorphism $C_0(\Omega) \to \mathcal{B}(\mathcal{A}_\hbar)$ associating to $\varphi \in C_0(\Omega)$ the operator $M_\varphi$, each $T_\hbar$ is contractive and $*$-linear. Since, by Theorem 4.3, the operator $Q_\hbar$ for $\hbar \in (0, 1)$ may be realized as $f(D_\hbar)$ for some $f \in C_0(\mathbb{R})$, standard arguments ([Gue98, Lem. 3.5], for example) show that $T_\hbar(\varphi)$ is a compact operator.

We have shown that for $\varphi \in C_0(\Omega)$ the family $\mathcal{B}(\varphi)$ defines a bounded section of the continuous field $\{ \mathcal{K}_\hbar \}$ depending $*$-linearly on $\varphi \in C_0(\Omega)$. It remains to prove that

(i) for $\varphi \in C_0(\Omega)$, $\mathcal{B}(\varphi)$ is a continuous section, and
(ii) $\mathcal{B}$ satisfies the asymptotic multiplicativity axiom.

For (i) we use the characterization of continuous sections of the field $\{ \mathcal{K}_\hbar \}$ given in Proposition 5.6. Since the unitary operators $U_\hbar : \mathcal{A}_\hbar \to \mathcal{A}$ are themselves multiplication operators they conjugate a multiplication operator on $\mathcal{A}$ to one on $\mathcal{A}_\hbar$.

Therefore, viewing the operators $U_\hbar$ as isometries, we have a commutative diagram

$$\begin{array}{ccc} H^2_\hbar(\Omega) & \xrightarrow{U_\hbar} & \mathcal{A} \\ {\scriptstyle T_\hbar(\varphi)}\Big\downarrow & & \Big\downarrow{\scriptstyle P_\hbar M_\varphi P_\hbar} \\ H^2_\hbar(\Omega) & \xrightarrow{U_\hbar} & \mathcal{A}. \end{array}$$

Since the family of projections $P_\hbar$ on $\mathcal{A}$ is norm continuous, the family of conjugates

$$U_\hbar T_\hbar(\varphi) U_\hbar^* = P_\hbar M_\varphi P_\hbar$$

is a norm continuous family of compact operators on $\mathcal{A}$.

We turn finally to (ii), the asymptotic multiplicativity. We use not only the fact that the operators $D_\hbar$ have gaps in their spectra (to write projections as continuous spectral functions of these operators as in (13)) but also the fact that these gaps become wider as $\hbar \to 0$. We must show that

$$\|T_\hbar(\varphi\psi) - T_\hbar(\varphi)T_\hbar(\psi)\| \to 0, \quad \text{as } \hbar \to 0, \tag{18}$$

the norms being taken in $\mathcal{B}(H^2_\hbar(\Omega))$. We work with the conjugated operators $P_\hbar M_\varphi P_\hbar$ on $\mathcal{A}$. From the simple calculation

$$\|P_\hbar M_\varphi M_\psi P_\hbar - P_\hbar M_\varphi P_\hbar M_\psi P_\hbar\| \leq \|P_\hbar M_\varphi - M_\varphi P_\hbar\| \|M_\psi\|$$

it follows that it suffices to show that

$$\|[P_\hbar, M_\varphi]\| \to 0, \quad \text{as } \hbar \to 0, \tag{19}$$

where $P_\hbar$ is the projection onto the kernel of $D_\hbar$. Let $f \in C_0(\mathbb{R})$ be supported in $[-1,1]$ and satisfy $f(0) = 1$. Let $s(\hbar)$ be a continuous function increasing to infinity as $\hbar \to 0$ and such that $s(\hbar) \leq 2\sqrt{1/\hbar - 1}$ for all $0 < \hbar \leq 1/2$ (for example, $s(\hbar) = 1/\sqrt{\hbar}$). The functions

$$f_\hbar(x) = f(s^{-1}x),$$

are supported in $[-s, s]$ and satisfy $f_\hbar(0) = 1$. Hence, by the properties of the spectra of $D_\hbar$ outlined in Theorem 4.3, we have $P_\hbar = f_\hbar(D_\hbar)$. The proof concludes with the observation that for all $g \in C_0(\mathbb{R})$ we have

$$\|[g_\hbar(D_\hbar), M_\varphi]\| \to 0, \quad \text{as } \hbar \to 0.$$

This is proved by observing that the set of such $g \in C_0(\mathbb{R})$ is a $*$-subalgebra which, by virtue of the fact that $s \to \infty$ as $\hbar \to 0$ and the calculation

$$\|[r_\pm(s^{-1}D_\hbar), M_\varphi]\| \le s^{-1}\|[M_\varphi, D_\hbar]\| \le s^{-1}\|\operatorname{grad}\varphi\|,$$

contains the resolvent functions $r_\pm(x) = \big(x \pm \sqrt{-1}\big)^{-1}$. □

We close with a few remarks regarding a result of Borthwick-Lesniewski-Upmeier [BLU93, Thm. 2.2] which states that (18) holds for continuous and bounded functions $\varphi$ and $\psi$, one of which is compactly supported. In the course of the proof of Theorem 5.7 we have proven the following generalization of their result, which for clarity we restate as

**Theorem 5.8.** *Let $\varphi$ and $\psi$ be continuous bounded functions on $\Omega$. Assume that $\varphi$ has a continuous extension to $\overline{\Omega}$. Then*

$$T_\hbar(\varphi)T_\hbar(\psi) - T_\hbar(\varphi\psi) \to 0, \quad \textit{as } \hbar \to 0,$$

*the norm being of bounded operators on the respective quantization spaces $H^2_\hbar(\Omega)$.*

*Proof.* The crux of the argument given above is that (19) holds provided $\varphi$ is continuously differentiable on $\Omega$ with bounded gradient. This clearly holds for $\varphi$ continuously differentiable on a neighborhood of $\overline{\Omega}$. Finally, if $\varphi$ is continuous on $\overline{\Omega}$ it can be approximated in the uniform norm by continuously differentiable $\varphi$. □

## 6. The Equality

The purpose of this section is to prove our second main theorem; the $E$-theory class of the Berezin quantization defined in the previous section is equal to the $E$-theory class of the Dolbeault operator of $\Omega$. We freely employ the notations of Sections 4 and 5. In particular,

$$\mathcal{A} = \bigoplus_q \mathcal{A}^{0q}, \quad \text{and} \quad \mathcal{A}_\hbar = \bigoplus_q \mathcal{A}^{0q}(E_\hbar).$$

are the Hilbert spaces of $E$ and $E_\hbar$-valued differential forms on $\Omega$ and

$$D : \mathcal{A} \to \mathcal{A}, \quad \text{and} \quad D_\hbar : \mathcal{A}_\hbar \to \mathcal{A}_\hbar$$

are ordinary Dolbeault operator and the Dolbeault operator twisted by $E_\hbar$. Recall that via the unitary isomorphisms $U_\hbar : \mathcal{A}_\hbar \to A$ the twisted Dolbeault operator $D_\hbar$ is unitary equivalent to the operator $D + \frac{1}{2\hbar}V$ on $\mathcal{A}$ where $V$ is the potential introduced in Section 4. By abuse of notation we denote this operator by $D_\hbar$ as well.

In order to define the $E$-homology class associated to the Dolbeault operator we require one additional piece of structure; the Hilbert spaces $\mathcal{A}$ and $\mathcal{A}_\hbar$ are graded by the decomposition into the spaces of even and odd forms. Denote by $\gamma$ the grading operator. With these notations established we recall that the $E$-homology class

$$[D] \in E_0(\Omega)$$

of the Dolbeault operator of $\Omega$ is defined to be the homotopy class of the asymptotic morphism

$$C_0(\mathbb{R}) \otimes C_0(\Omega) \to C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{A})$$

defined on basic tensors by

$$f \otimes \varphi \longmapsto f(t^{-1}D + x\gamma)\varphi, \quad \text{for all } f \in C_0(\mathbb{R}) \text{ and } \varphi \in C_0(\Omega).$$

This construction appeared in the original unpublished manuscript of Connes and Higson [CH89]; for details of the construction we refer to [Gue98].

**Theorem 6.1.** *The $E$-homology classes of the Berezin quantization and Dolbeault operator are equal:*

$$[\mathcal{B}] = [D] \in E_0(\Omega).$$

*Remark.* The classes $[\mathcal{B}]$ and $[D]$, and in particular the $E$-homology group $E_0(\Omega)$, are nonzero. We are unable to find a reference for this elementary fact in the literature (but compare [BD82]), so provide the following simple argument.

Extension by zero defines a $*$-homomorphism $C_0(\Omega) \to C_0(\mathbb{C}^n)$ which induces a homomorphism $E_0(\mathbb{C}^n) \to E_0(\Omega)$. This is an isomorphism. Indeed, if $B \subset \Omega$ is a small ball we similarly have $E_0(\Omega) \to E_0(B)$ and the composite map $E_0(\mathbb{C}^n) \to E_0(B)$ is an isomorphism; since all groups in question are isomorphic to $\mathbb{Z}$ the result follows.

To complete the argument recall that the class of the Dolbeault operator on $\mathbb{C}^n$ is nonzero in $E_0(\mathbb{C}^n)$; indeed it generates $E_0(\mathbb{C}^n) \cong \mathbb{Z}$ (compare [Ati68]). Further, it follows from [Gue99a] that its image under the map $E_0(\mathbb{C}^n) \to E_0(\Omega)$ is $[D] \in E_0(\Omega)$.

We proceed to the proof of Theorem 6.1 which will occupy the remainder of the section. Along the way we encounter the $E$-homology classes of a number of other asymptotic morphisms. We mention two explicitly.

**Proposition 6.2.** *The family of functions $\alpha_t$ defined by (we write $\hbar = t^{-1}$),*

$$\alpha_t(f \otimes \varphi) = f(D_\hbar + x\gamma)\varphi, \quad \textit{for all } f \in C_0(\mathbb{R}) \textit{ and } \varphi \in C_0(\Omega)$$

*extends to an asymptotic morphism $\alpha_t$ from $C_0(\mathbb{R}) \otimes C_0(\Omega)$ to $C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{A})$. Furthermore, $\alpha_t$ represents the $E$-homology class of the Berezin quantization:*

$$[\alpha_t] = [\mathcal{B}] \in E_0(\Omega).$$

*Proof.* We sketch the proof that $\alpha_t$ defines an asymptotic morphism, following closely the proof of Theorem 3.4 of [Gue98]. We must show

(i) $f \longmapsto f(D_\hbar + x\gamma)$ defines a continuous family of $*$-homomorphisms from $C_0(\mathbb{R})$ to $C_0(\mathbb{R}, \mathcal{B}(\mathcal{A}))$, and
(ii) $[\varphi, f(D_\hbar + x\gamma)] \to 0$ as $\hbar \to 0$, for $f \in C_0(\mathbb{R})$ and $\varphi \in C_0(\Omega)$.

The proof of (i) is slightly easier than in [Gue98] by virtue of the fact that the identity

$$r_\pm(D_\hbar + x\gamma) - r_\pm(D_{\hbar'} + x\gamma) = r_\pm(D_\hbar + x\gamma)\left(\frac{V}{2}(\frac{1}{\hbar'} - \frac{1}{\hbar})\right) r_\pm(D_{\hbar'} + x\gamma),$$

is simpler than its counterpart in [Gue98]. The proof of (ii) is somewhat more difficult and is accomplished by decomposing the operator $f(D_\hbar + x\gamma)$ with respect to the decomposition of $\mathcal{A}$ into $\ker(D_\hbar)$ and its orthogonal complement. For $f \in C_0(\mathbb{R})$ we have

$$f(D_\hbar + x\gamma) - f(x)P_\hbar \to 0, \quad \text{as } \hbar \to 0 \tag{20}$$

by virtue of the spectral properties of $D_\hbar$ outlined in Theorem 4.3. The result follows easily since we observed in the proof of Theorem 5.7 that

$$\|[P_\hbar, M_\varphi]\| \to 0, \quad \text{as } \hbar \to 0. \tag{21}$$

We now turn to the equality in the statement of the proposition. The zero element of $E_0(\Omega)$ is represented by the zero homomorphism $C_0(\Omega) \to \mathcal{K}(H)$. Recall that the family of projections $P_\hbar$ is a norm continuous family, and hence the same is true of the projections $1 - P_\hbar$. Thus, by Lemma 7.1 the zero element of $E_0(\Omega)$ is represented by the zero generalized asymptotic morphism $C_0(\Omega) \to C_b(\{\, \mathcal{K}((1-P_\hbar)\mathcal{A})\, \})$. Adding this class to the Berezin class we see that the latter is represented by the asymptotic morphism

$$\varphi \longmapsto P_\hbar M_\varphi P_\hbar : C_0(\Omega) \to \mathcal{K}(\mathcal{A}).$$

Further, by (21) this asymptotic morphism is asymptotically equivalent to the asymptotic morphism

$$\varphi \longmapsto P_\hbar M_\varphi : C_0(\Omega) \to \mathcal{K}(\mathcal{A}).$$

But, the suspension of this asymptotic morphism is in turn asymptotically equivalent to $\alpha_t$ by (20). □

**Proposition 6.3.** *The family of functions $\beta_t$ defined on basic tensors by*

$$\beta_t(f \otimes \varphi) = f(t^{-1/4}(D + V/2) + x\gamma), \quad \text{for all } f \in C_0(\mathbb{R}) \text{ and } \varphi \in C_0(\Omega)$$

*extends to an asymptotic morphism from $C_0(\mathbb{R}) \otimes C_0(\Omega)$ to $C_0(\mathbb{R}) \otimes \mathcal{K}(\mathcal{A})$. Furthermore, $\beta_t$ represents the $E$-homology class of the Dolbeault operator of $\Omega$:*

$$[\beta_t] = [D] \in E_0(\Omega).$$

*Proof.* The proof that $\beta_t$ defines an asymptotic morphism is identical to the proof of Theorem 3.4 in [Gue98]. In fact, $\beta_t$ is a simple rescaling of the asymptotic morphism defining the $E$-homology class of the operator $D + V/2$ and hence represents that class. But there is an equality

$$[D] = [D + V/2] \in E_0(\Omega)$$

([Gue98], Proposition 3.7) and the desired result follows. □

With these propositions in hand we can complete the proof of Theorem 6.1:

*Proof of Theorem 6.1.* In the notation of the previous two propositions it suffices to show that the asymptotic morphisms $\alpha_t$ and $\beta_t$ from $C_0(\mathbb{R})\otimes C_0(\Omega)$ to $C_0(\mathbb{R})\otimes\mathcal{K}(\mathcal{A})$ represent the same $E$-homology class:

$$[\alpha_t] = [\beta_t] \in E_0(\Omega).$$

We construct an explicit homotopy from $\alpha_t$ to $\beta_t$. Define a family of functions $\eta_t$ by

$$\eta_t(f\otimes\varphi) = f\left(\sigma^{-1}D_{\tau^{-1}} + x\gamma\right)\varphi, \quad \text{for all } f\in C_0(\mathbb{R}) \text{ and } \varphi\in C_0(\Omega),$$

where we have defined for $s\in[0,1]$ and $t\geq 1$

$$\sigma^4 = ((1-s)+st)\,, \quad \text{and} \quad \tau = ((1-s)t+s)\,.$$

It is immediate from these definitions that $\alpha_t$ and $\beta_t$ are obtained by composing $\eta_t$ with evaluation at $s=0$ and $s=1$, respectively. It remains only to check that $\eta_t$ defines an asymptotic morphism from $C_0(\mathbb{R})\otimes C_0(\Omega)$ to $C_0(\mathbb{R}\times[0,1],\mathcal{K}(\mathcal{A}))$. This follows from Lemma 7.1 of [Gue98] once we prove:

(i) $\varphi\longmapsto 1\otimes 1\otimes M_\varphi$ is a $*$-homomorphism from $C_0(\Omega)$ into $C_b(\mathbb{R}\times[0,1],\mathcal{B}(\mathcal{A}))$,
(ii) $f\longmapsto f\left(\sigma^{-1}D_{\tau^{-1}}+x\gamma\right)$ is a continuous family of $*$-homomorphisms from $C_0(\mathbb{R})$ to $C_0(\mathbb{R}\times[0,1],\mathcal{B}(\mathcal{A}))$,
(iii) for fixed $s\in[0,1]$ and $t\geq 1$ the product $f\left(\sigma^{-1}D_{\tau^{-1}}+x\gamma\right)M_\varphi$ is a compact operator on $\mathcal{A}$, and
(iv) for all $f\in C_0(\mathbb{R})$ and $\varphi\in C_0(\Omega)$ the commutator $[M_\varphi, f\left(\sigma^{-1}D_{\tau^{-1}}+x\gamma\right)]$ tends to zero as $t\to\infty$ (considered as an element of $C_0(\mathbb{R}\times[0,1],\mathcal{K}(\mathcal{A}))$).

Of these, (i) is obvious, (iii) follows from standard arguments (see the proof of Theorem 5.7), and (ii) ane (iv) are treated in the following lemmas. ☐

**Lemma 6.4.** *The assignment*

$$f\longmapsto f\left(\sigma^{-1}D_{\tau^{-1}}+x\gamma\right), \quad \textit{for all } f\in C_0(\mathbb{R}) \tag{22}$$

*defines a continuous family of $*$-homomorphisms from $C_0(\mathbb{R})$ to $C_0(\mathbb{R}\times[0,1],\mathcal{B}(\mathcal{A}))$.*

*Proof.* We must verify that for fixed $t \geq 1$ the expression in (22) is continuous in $(x,s) \in \mathbb{R} \times [0,1]$ and vanishes at infinity. Further, we must verify that the resulting element of $C_0(\mathbb{R} \times [0,1], \mathcal{B}(\mathcal{A}))$ varies continuously with $t \geq 1$. It suffices to consider the case $f = r_\pm$ is one of the resolvent functions.

By virtue of the identity $(aD_b + x\gamma)^2 = a^2D_b^2 + x^2$ we see that the spectrum of $aD_b + x\gamma$ lies in the complement of $(-x,x)$, independently of $a$ and $b \in \mathbb{R}$. Thus, simple calculations show that

$$\|r_\pm\,(aD_b + x\gamma)\,\| \leq \frac{1}{\sqrt{x^2+1}}, \tag{23}$$

independently of $a$ and $b \in \mathbb{R}$. From this we conclude that for each $t \geq 1$ the resolvent $r_\pm(\sigma^{-1}D_{\tau^{-1}} + x\gamma)$ vanishes at infinity in $(x,s) \in \mathbb{R} \times [0,1]$. We record for later use the similar fact that, independently of $a$, $b$, and $x \in \mathbb{R}$,

$$\|\,(aD_b + x\gamma)\,r_\pm\,(aD_b + x\gamma)\,\| \leq 1. \tag{24}$$

Define for $s' \in [0,1]$ and $t' \geq 1$

$$\sigma_1^4 = ((1-s') + s't')\,, \quad \text{and} \quad \tau_1 = ((1-s')t' + s')\,, \tag{25}$$

and introduce the convenient shorthand

$$\mathbb{D} = \sigma^{-1}D_{\tau^{-1}} + x\gamma, \quad \text{and} \quad \mathbb{D}_1 = \sigma_1^{-1}D_{\tau_1^{-1}} + x_1\gamma.$$

From the resolvent identity

$$r_\pm(\mathbb{D}) - r_\pm(\mathbb{D}_1) = r_\pm(\mathbb{D}_1)\,(\mathbb{D} - \mathbb{D}_1)\,r_\pm(\mathbb{D}). \tag{26}$$

we are lead to calculate the difference $\mathbb{D} - \mathbb{D}_1$. Since

$$D_{\tau^{-1}} - D_{\tau_1^{-1}} = \left(D + \frac{\tau}{2}V\right) - \left(D + \frac{\tau_1}{2}V\right) = \left(\frac{\tau - \tau_1}{2}\right)V.$$

we have

$$\begin{aligned}
\sigma^{-1}D_{\tau^{-1}} \quad - \quad \sigma_1^{-1}D_{\tau_1^{-1}} &= \sigma^{-1}D_{\tau^{-1}} - \sigma_1^{-1}\left(D_{\tau^{-1}}\frac{\tau-\tau_1}{2}V\right) \\
&= (\sigma^{-1}-\sigma_1^{-1})D_{\tau^{-1}} + \frac{\sigma_1^{-1}(\tau-\tau_1)}{2}V \\
&= (1-\sigma\sigma_1^{-1})\sigma^{-1}D_{\tau^{-1}} + \frac{\sigma_1^{-1}(\tau-\tau_1)}{2}V \\
&= (1-\sigma\sigma_1^{-1})\left(\sigma^{-1}D_{\tau^{-1}}+x\gamma\right) - (1-\sigma\sigma_1^{-1})x\gamma + \frac{\sigma_1^{-1}(\tau-\tau_1)}{2}V \\
&= (1-\sigma\sigma_1^{-1})\mathbb{D} - (1-\sigma\sigma_1^{-1})x\gamma + \frac{\sigma_1^{-1}(\tau-\tau_1)}{2}V.
\end{aligned}$$

We therefore conclude that

$$\mathbb{D}-\mathbb{D}_1 = (1-\sigma\sigma_1^{-1})\mathbb{D} - (1-\sigma\sigma_1^{-1})x\gamma + \frac{\sigma_1^{-1}(\tau-\tau_1)}{2}V + (x-x_1)\gamma. \tag{27}$$

Collecting (26) and (27) we conclude that for all $x\in\mathbb{R}$, $s\in[0,1]$ and $t\geq 1$ the norm of $r_\pm(\mathbb{D})-r_\pm(\mathbb{D}_1)$ is bounded by the sum of four terms

$$\begin{gathered}
|1-\sigma\sigma_1^{-1}|\,\|r_\pm(\mathbb{D}_1)\|\,\|\mathbb{D}r_\pm(\mathbb{D})\| \\
|1-\sigma\sigma_1^{-1}|\,|x|\,\|r_\pm(\mathbb{D}_1)\|\,\|r_\pm(\mathbb{D})\| \\
\left|\tfrac{\sigma_1^{-1}(\tau-\tau_1)}{2}\right|\,\|V\|\,\|r_\pm(\mathbb{D}_1)\|\,\|r_\pm(\mathbb{D})\| \\
|x-x_1|\,\|r_\pm(\mathbb{D}_1)\|\,\|r_\pm(\mathbb{D})\|
\end{gathered}$$

Employing now (23) and (24) we bound the sum of these four terms, and hence the norm of $r_\pm(\mathbb{D})-r_\pm(\mathbb{D}_1)$ by

$$\|r_\pm(\mathbb{D})-r_\pm(\mathbb{D}_1)\| \leq C_1|1-\sigma\sigma_1^{-1}| + C_2|\sigma_1^{-1}(\tau-\tau_1)| + K|x-x_1|, \tag{28}$$

where $C_1$, $C_2$ and $K$ are constants independent of $(x,s)\in\mathbb{R}\times[0,1]$ and $t\geq 1$.

To conclude the proof we require some further elementary estimates. We have

$$|\tau-\tau_1| \leq (1+t)|s'-s| + |t'-t|, \tag{29}$$

and a similar estimate for $|\sigma^4 - \sigma_1^4|$. Further, $\sigma^4 = (1-s) + st \geq 1$ so that $\sigma \geq 1$ and $\sigma^{-1} \leq 1$. We conclude that

$$|\sigma^4 - \sigma_1^4| = |\sigma^2 - \sigma_1^2|\,|\sigma^2 + \sigma_1^2| \geq 2|\sigma - \sigma_1|\,|\sigma + \sigma_1| \geq 4|\sigma - \sigma_1|,$$

and further that

$$4|1 - \sigma\sigma_1^{-1}| = 4|\sigma_1 - \sigma|\,|\sigma_1^{-1}| \leq (1+t)|s' - s| + |t' - t|. \tag{30}$$

For our final estimate we combine (28), (29) and (30) to obtain a bound on the norm of $r_\pm(\mathbb{D}) - r_\pm(\mathbb{D}_1)$ in terms of $(x,s) \in \mathbb{R} \times [0,1]$ and $t \geq 1$:

$$\|r_\pm(\mathbb{D}) - r_\pm(\mathbb{D}_1)\| \leq C\left((1+t)|s' - s| + |t' - t|\right) + K|x - x_1|,$$

where $C$ and $K$ are constants independent of $(x,s) \in \mathbb{R} \times [0,1]$ and $t \geq 1$. The content of this last estimate is that for a fixed $T \geq 1$ the resolvent $r_\pm(\mathbb{D})$ is uniformly continuous in the variables $(x,s) \in \mathbb{R} \times [0,1]$ and $t \in [1,T]$. The desired result follows. □

**Lemma 6.5.** *For all $f \in C_0(\mathbb{R})$ and $\varphi \in C_0(\Omega)$ the commutator*

$$[M_\varphi, f\left(\sigma^{-1} D_{\tau^{-1}} + x\gamma\right)], \tag{31}$$

*tends to zero as an element of $C_0(\mathbb{R} \times [0,1], \mathcal{K}(\mathcal{A}))$ as $t \to \infty$.*

*Proof.* The proof of this proposition has much in common with the proof of Proposition 6.2. In particular, we rely on the fact that for all $\varphi \in C_0(\Omega)$

$$[M_\varphi, P_{t^{-1}}] \to 0, \quad \text{as } t \to \infty \tag{32}$$

(compare to (21)). Our task is to show that the norm of the commutator (31) tends to zero as $t \to \infty$ uniformly in $(x,s) \in \mathbb{R} \times [0,1]$. We consider the cases $s \leq 1/2$ and $s \geq 1/2$ separately, beginning with the former.

Retaining the notation of the previous proof, simple calculations show that for all $f \in C_0(\mathbb{R})$ and $\varphi \in C_0(\Omega)$

$$\|[M_\varphi, f(\mathbb{D})\| \leq 2\|\varphi\|\|f \otimes P_{\tau^{-1}} - f(\mathbb{D})\| + \|[M_\varphi \otimes 1, P_{\tau^{-1}} \otimes f]\|. \tag{33}$$

We proceed as in the proof of Proposition 6.2 to estimate the norm of the difference

$$f(x)P_{\tau^{-1}} - f(\mathbb{D}) = f(x)P_{\tau^{-1}} - f\left(\sigma^{-1}D_{\tau^{-1}} + x\gamma\right) \tag{34}$$

by breaking the space $\mathcal{A}$ into the direct sum of $P_{\tau^{-1}}\mathcal{A} = \ker D_{\tau^{-1}}$ and its orthogonal complement. On $\ker D_{\tau^{-1}}$ this difference is zero. On its orthogonal complement $P_{\tau^{-1}}$ is zero and the spectral properties of the operators $D_{\tau^{-1}}$ outlined in Theorem 4.3 imply that

$$\left(\sigma^{-1}D_{\tau^{-1}} + x\gamma\right)^2 = \sigma^{-2}D^2_{\tau^{-1}} + x^2 \geq \sigma^{-2}D^2_{\tau^{-1}} \geq 4\sigma^{-2}(\tau - 1),$$

independently of $x \in \mathbb{R}$. We conclude that the norm of (34) is bounded by

$$\sup\{\,|f(y)| : |y| \geq 2\sigma^{-1}\sqrt{\tau - 1}\,\},$$

independently of $x \in \mathbb{R}$. To show that this expression tends to zero uniformly in $s \in [-, 1/2]$ as $t \to \infty$ we show that the expression $2\sigma^{-1}\sqrt{\tau - 1}$ tends to infinity uniformly in $s \in [0, 1/2]$ as $t \to \infty$. This follows from simple estimates regarding $\tau$ and $\sigma$, each independent of $s \in [0, 1/2]$:

(i) $\sigma^{-1} \geq t^{-1/4}$
(ii) $4(\tau - 1) \geq 2(t - 1) \geq t$, for $t \geq 2$.

Estimate (i) follows from $\sigma^4 = (1 - s) + st \leq t$ and (ii) from $\tau - 1 = (t - 1)(1 - s)$. Combining (i) and (ii) we conclude that for $t \geq 2$,

$$2\sigma^{-1}\sqrt{\tau - 1} \geq t^{1/4},$$

independently of $s \in [0, 1/2]$. For such $s$ we also have $\tau = (1 - s)t + s \geq t/2$, so that it follows from (32) that

$$[M_\varphi, P_{\tau^{-1}}] \to 0, \quad \text{as } t \to \infty$$

uniformly in $s \in [0, 1/2]$. Combining what we have thus far, we have shown that the norm of (33) tends to zero as $t \to \infty$ uniformly in $(x, s) \in \mathbb{R} \times [0, 1/2]$.

We now turn our attention to the complementary interval $s \in [1/2, 1]$. On this interval we reduce to consideration of the resolvent functions $f = r_\pm$ and smooth and compactly supported $\varphi$. We use the identity

$$[M_\varphi, r_\pm(\mathbb{D})] = r_\pm(\mathbb{D})[\mathbb{D}, M_\varphi]r_\pm(\mathbb{D}) = r_\pm(\mathbb{D})\sigma^{-1}[D_{\tau^{-1}}, M_\varphi]r_\pm(\mathbb{D}), \tag{35}$$

from which follows that

$$\|[M_\varphi, r_\pm(\mathbb{D})]\| \leq \sigma^{-1}\|\,\text{gradient of } \varphi\,\|.$$

It thus remains only to verify that $\sigma \to \infty$ uniformly in $s \in [1/2, 1]$ as $t \to \infty$. It is, however, immediate from the definition of $\sigma$ that $\sigma^4 \geq t/2$ independently of $s \in [1/2, 1]$. The lemma is thereby proved. □

## 7. Appendix: Continuous Fields

In establishing notation and conventions for continuous fields of Hilbert spaces and $C^*$-algebras we follow Dixmier [Dix70, Ch. 10].

A *continuous field* of Hilbert spaces over a topological space $T$ consists of a family of Hilbert spaces $H_t$, $t \in T$ together with a vector space $\Gamma$ of functions $x(t)$ satisfying certain axioms [Dix70, 10.1.2]. We denote a continuous field by $(\{\,H_t\,\}, \Gamma)$, although whenever convenient we shall omit $\Gamma$ from the notation, denoting the continuous field by $\{\,H_t\,\}$. If each $H_t$ is equal to a fixed Hilbert space $H$ and $\Gamma$ is the set of continuous functions on $T$ with values in $H$ the field is called *constant*. We shall consistently denote the constant field by $\{\,H\,\}$. A field isomorphic to a constant field is called *trivial*.

**Lemma 7.1.** *Let $H$ be Hilbert space. For all $t$ in a locally compact topological space $T$ let $H_t$ be a Hilbert space and $U_t : H_t \to H$ be an isometry. Assume that the family of range projections $P_t$ of $U_t$ is strongly continuous. The collection*

$$\Gamma = \{\,x(t) \in H_t : U_t x(t) \text{ is continuous}\,\}$$

*defines the structure of a continuous field $\{\,H_t\,\}$. Further, if $T$ is an interval and the $P_t$ are norm continuous the field is trivial.*

*Proof.* To see that $\Gamma$ defines the structure of a continuous field we verify the axioms directly. It is clear that $\Gamma$ is a linear space and that for $x \in \Gamma$ the function $\|x(t)\|$

is continuous. To show that $\{ x(s) : x \in \Gamma \} = H_s$ let $v \in H_s$ be given. Define $w = U_s(v) \in H$ and $x(t) = U_t^* w$. It is easily verified that $x(s) = v$ and further that $U_t x(t) = P_t w$ is continuous in $t$. Finally let $y(t) \in H_t$ be a function that is the local uniform limit of elements of $\Gamma$. To see that $y \in \Gamma$ let $s$ be given and show that $U_t y(t)$ is continuous at $s$. Let $\varepsilon > 0$ be given and obtain an open neighborhood $O$ of $s$ in $T$ and an $x \in \Gamma$ such that

$$\|y(t) - x(t)\| < \varepsilon, \quad \text{for all } t \in O.$$

By reducing to a smaller neighborhood of $s$ if necessary we further arrange that

$$\|U_t x(t) - U_s x(s)\| < \varepsilon, \quad \text{for all } t \in O.$$

It is then straightforward to verify that

$$\|U_t y(t) - U_s y(s)\| \leq 3\varepsilon, \quad \text{for all } t \in O.$$

We turn to the triviality of the field in the case that $T$ is an interval and the $P_t$ are norm continuous. In this case there exists a norm continuous family of unitaries $V_t$ such that for all $t$

$$V_t P_t V_t^* = P_1 = P.$$

It follows that $PH = V_t P_t V_t^* H = V_t P_t H$ and we therefore may view the product $V_t U_t$ as a unitary operator

$$H_t \xrightarrow{U_t} P_t H \xrightarrow{V_t} PH.$$

This collection of unitary isomorphisms provides the desired trivialization; it is readily verified that $x \in \Gamma$ if and only if $V_t U_t x(t)$ is a continuous $H$-valued function. $\square$

**Lemma 7.2.** *Employ the notation of Lemma 7.1. Let $\mathcal{K}_t$ be the $C^*$-algebra of compact operators on $H_t$ and $(\{ K_t \}, \Gamma')$ the continuous field of elementary $C^*$-algebras associated to $\{ H_t \}$. The collection of continuous sections of $\{ \mathcal{K}_t \}$ is*

$$\Gamma'' = \{ K(t) \in \mathcal{K}_t : U_t K(t) U_t^* \textit{ is continuous} \}.$$

*Proof.* Is suffices to show that $\Gamma' \subset \Gamma''$ (compare [Dix70]). Recall that $\Gamma'$ is the closure with respect to local uniform convergence of the linear span of the rank one families associated to continuous sections of $\{ H_t \}$;

$$\Theta_{x,y}(t) = \Theta_{x(t),y(t)} = \langle \cdot, x(t) \rangle y(t).$$

It therefore suffices to show that

(i) $\Theta_{x,y} \in \Gamma''$ for all $x$ and $y \in \Gamma$, and
(ii) the collection $\Gamma''$ is closed under local uniform convergence.

The first assertion follows from the simple estimate

$$\|\Theta_{x,y} - \Theta_{x',y'}\| \leq \|x\|\|y - y'\| + \|x - x'\|\|y'\|$$

and calculation

$$U_t \Theta_{x,y}(t) U_t^* = \Theta_{U_t x(t), U_t y(t)}.$$

The second assertion follows as in the proof of the previous lemma. □

## 8. Appendix: $E$-Theory

Let $A$ and $B$ be $C^*$-algebras. An *asymptotic morphism* from $A$ to $B$ is a family of functions $\{\varphi_t\} : A \to B$ indexed by $t \in T = [1, \infty)$ satisfying the continuity condition

$$t \mapsto \varphi_t(a) \text{ is a continuous } B\text{-valued function for all } a \in A$$

as well as the asymptotic conditions

$$\lim_{t\to\infty} \left\{ \begin{array}{c} \varphi_t(a) + \lambda\varphi_t(a') - \varphi_t(a + \lambda a') \\ \varphi_t(a)\varphi_t(a') - \varphi_t(aa') \\ \varphi_t(a)^* - \varphi_t(a^*) \end{array} \right\} = 0, \qquad \text{for all } a, a' \in A \text{ and } \lambda \in \mathbb{C}.$$

Asymptotic morphisms $\{\varphi_t\}$ and $\{\psi_t\}$ are *asymptotically equivalent* if

$$\lim_{t\to\infty} (\varphi_t(a) - \psi_t(a)) = 0, \quad \text{for all } a \in A.$$

Denote by $B[0,1]$ the $C^*$-algebra of continuous $B$-valued functions on the closed interval $[0,1]$. Asymptotic morphisms $\{\varphi_t\}$ and $\{\psi_t\}$ are *homotopic* if there is an asymptotic morphism $\{\alpha_t\} : A \to B[0,1]$ from which they may be recovered upon composition with evaluation at zero and one. Both asymptotic equivalence and homotopy are equivalence relations on the set of asymptotic morphisms from $A$ to $B$. The set of homotopy classes is denoted $[\![A, B]\!]$.

Let $\mathcal{K}$ be the $C^*$-algebra of compact operators on a separable, infinite dimensional Hilbert space. Let $\mathcal{S}$ be the $C^*$-algebra of continuous functions on $\mathbb{R}$ vanishing at infinity and for any $C^*$-algebra $A$ denote the *suspension* of $A$ by $\mathcal{S}A = \mathcal{S} \otimes A$.

The *bivariant $E$-theory groups* are defined by $E(A, B) = [\![\mathcal{S}A, \mathcal{S}B \otimes \mathcal{K}]\!]$. Our primary concern is with the *$E$-homology* groups defined by

$$E^0(A) = E(A, \mathbb{C}) = [\![\mathcal{S}A, \mathcal{S} \otimes \mathcal{K}]\!],$$

although when speaking about commutative $C^*$-algebras, $A = C_0(X)$ where $X$ is a locally compact metrizable space, it is customary to denote these groups by

$$E_0(X) = E^0(C_0(X)).$$

*Remark.* There are many seemingly different, but nonetheless equivalent, versions of $E$-theory [Dad94, Gue99b, GHT00]. The equivalence of our definition with the original definition of Connes-Higson is proven by a slight adaptation of the arguments in [Hig87]. The equivalence of our definition with the one employed in [Gue98, Gue99a] follows immediately from Bott Periodicity (compare [CH89]); we will use several results from these references.

In defining the $E$-theory class associated to the Berezin quantization it is convenient to use a slightly generalized notion of asymptotic morphism. The benefit of this slightly generalized notion is primarily one of notational convenience.

Let $\{ H_\hbar \}$ be a continuous field of Hilbert spaces on the interval $\hbar \in (0, 1)$, together with a trivialization $\{ U_\hbar \}$. In particular, the $U_\hbar$ are unitary isomorphisms from the $H_\hbar$ to a fixed Hilbert space $H$ and the continuous sections of $\{ H_\hbar \}$ are precisely the translates of continuous functions of $\hbar$ with values in $H$. The associated field of elementary $C^*$-algebras, denoted $\mathcal{K}(\{ H_\hbar \})$ is trivialized by $\{ \operatorname{ad} U_\hbar \}$, where $\operatorname{ad} U_\hbar :$

$\mathcal{K}(H_\hbar) \to \mathcal{K}(H)$ is conjugation with $U_\hbar$. Denote by $C_b(\{\,\mathcal{K}_\hbar\,\})$ and $C_0(\{\,\mathcal{K}_\hbar\,\})$ the $C^*$-algebras of continuous bounded sections and continuous sections vanishing at zero of $\mathcal{K}(\{\,H_\hbar\,\})$, respectively.

A *generalized asymptotic morphism* is a function $\varphi$ from $A$ into the set of sections of $\mathcal{K}\,(\{\,H_\hbar\,\})$ satisfying the continuity condition

$$\varphi(a) \in C_b(\{\,\mathcal{K}_\hbar\,\}) \text{ for all } a \in A$$

as well as the obvious asymptotic conditions

$$\left\{\begin{array}{c} \varphi(a) + \lambda\varphi(a') - \varphi(a+\lambda a') \\ \varphi(a)\varphi(a') - \varphi(aa') \\ \varphi(a)^* - \varphi(a^*) \end{array}\right\} \in C_0(\{\,\mathcal{K}_\hbar\,\}), \text{for all } a, a' \in A \text{ and } \lambda \in \mathbb{C}.$$

A generalized asymptotic morphism $\varphi$ gives an asymptotic morphism $\{\widetilde{\varphi}_t\}$ via the prescription

$$\widetilde{\varphi}_t(a) = \operatorname{ad} U_{1/t}(\varphi(a)(1/t)), \quad \text{for all } a \in A.$$

**Lemma 8.1.** *The homotopy class of the asymptotic morphism associated to the generalized asymptotic morphism $\varphi$ is independent of the chosen trivialization of the field $\{\,H_\hbar\,\}$.*

*Proof.* Let $\{\,U_\hbar\,\}$ be an isomorphism of constant fields $\{\,H\,\} \cong \{\,H\,\}$ and $\varphi : A \to C_b\{\,\mathcal{K}(H)\,\}$ be a generalized asymptotic morphism. We prove that the asymptotic morphisms associated to $\varphi$ and $U\varphi$ are homotopic, where

$$U\varphi(a)(\hbar) = U_\hbar(\varphi(a)(\hbar)), \quad \text{for all } a \in A \text{ and } \hbar \in (0,1].$$

A homotopy from $\widetilde{\varphi}_t$ to $(\widetilde{U\varphi})_t$ is defined by

$$\alpha_t(a)(s) = U_{st+(1-s)}\varphi_t(a)U^*_{st+(1-s)}, \quad \text{for all } a \in A \text{ and } s \in [0,1].$$

The required continuity properties of $\alpha_t$ follow from elementary facts about the unitary group $\mathcal{U}$ of $H$ equipped with the strong operator topology; $\mathcal{U}$ is a metrizable topological space and acts on $\mathcal{K}$ as a topological transformation group. In particular, the map

$$(U,K) \longmapsto UKU^* : \mathcal{U} \times \mathcal{K} \to \mathcal{K}$$

is continuous so that the function $U_r \varphi_t(s) U_r^*$ of $s \in [0,1]$ and $r \geq 1$ is uniformly continuous on the the product of $[0,1]$ with any compact initial segment of the ray $r \geq 1$. The asymptotic properties of $\alpha_t$ are also straightforwardly verified. □

We summarize the result from this appendix used in the defining the $E$-theory class of the Berezin quantization.

**Proposition 8.2.** *A generalized asymptotic morphism $\varphi : A \to C_b(\{\mathcal{K}_\hbar\})$ defines an element of the $E$-homology group $E(A, \mathbb{C})$. This element is independent of the choice of trivialization used to define it.* □

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