Proper actions and weak amenability of classical relatively hyperbolic groups

E. Guentner[*], E. Reckwerdt[†], R. Tessera[‡]

Abstract

Gromov introduced a notion of hyperbolicity for discrete groups (and general metric spaces) as an abstraction of the properties of universal covers of closed, negatively curved manifolds and their fundamental groups. The fundamental group of a manifold with pinched negative curvature and a cusp is not hyperbolic, but it is relatively hyperbolic with respect to the cusp subgroup, which has polynomial growth. We introduce a *thinning* technique which allows to reduce questions about these classical relatively hyperbolic groups to the case of bounded geometry hyperbolic graphs. As applications, we show that such groups admit a proper affine action on an L^p-space and are weakly amenable in the sense of Cowling-Haagerup. These results generalize earlier work of G. Yu and N. Ozawa, respectively, from the setting of hyperbolic groups to classical relatively hyperbolic groups.

1. Introduction

Many properties of hyperbolic groups seem natural to extend to relatively hyperbolic groups. A group is hyperbolic if it acts properly and cocompactly on a hyperbolic space. Roughly speaking, a group is hyperbolic relative to a subgroup if, *modulo that subgroup*, it acts properly on a hyperbolic space. See §2 for a precise definition.

Unfortunately, many properties of hyperbolic graphs (and metric spaces) rely on a bounded geometry assumption in their proofs. The most relevant examples for us are the existence of proper affine actions of hyperbolic groups on ℓ^p-spaces [Yu05], and weak amenability [Oza08], and each of these properties will be discussed below. Another example is finite asymptotic dimension [Roe05]. The hyperbolic spaces usually introduced to study relative hyperbolicity do not have bounded geometry,

[*]E. G. was supported in part by a grant from the Simons Foundation (#245398 and #586754).
[†]E. R. was supported in part by ERC grant no. 677120-INDEX.
[‡]R. T. was supported in part by ANR-14-CE25-0004 GAMME.

and some are not even locally finite. In generalizing results such as those above to the relatively hyperbolic setting a key step is to develop an appropriate substitute for bounded geometry. In the case of finite asymptotic dimension, for example, see [Osi05].

The concept of a relatively hyperbolic group was originally introduced by Gromov as a general setup which included non-uniform lattices in rank one simple Lie groups and, more generally, fundamental groups of cusped manifolds with pinched negative curvature. In these cases the parabolic, or cusp subgroups are nilpotent. Taking this as motivation, we shall call a group that is relatively hyperbolic with respect to subgroups of polynomial growth a *classical relatively hyperbolic group*. We shall see that this class is somehow easier to work with due to the following fact: a Cayley graph of a classical relatively hyperbolic group G can be embedded into a *bounded geometry* hyperbolic graph such that each coset of a parabolic subgroup lies at bounded distance from a horosphere. See section 3, especially Proposition 3.7. This hyperbolic graph, and its construction, would therefore seem perfectly suited to allow for generalizing properties of hyperbolic groups to classical relatively hyperbolic groups. However, a significant defect of this graph is that it is does *not* (and indeed cannot) admit an action of G. We do, however, have an action of G on the *space of all such graphs*, and our main result in this paper is to show that this space can be equipped with a G-invariant probability measure. See Proposition 3.6.

While we believe these ideas will be useful for future applications, in the present work we shall use them to adapt two results from the setting of hyperbolic to classical relatively hyperbolic groups: the first, due to Yu [Yu05], and building on a key averaging technique due to Mineyev [Min01], concerns proper affine actions; the second, due to Ozawa [Oza08] concerns weak amenability. Here then are our results.

Theorem A. *Let G be a finitely generated discrete group which is hyperbolic relative to a subgroup P of polynomial growth. Then G admits a (metrically) proper action on a mixed ℓ^p–ℓ^1-space, and also on an L^p-space, for sufficiently large p.*

Theorem B. *A discrete group G as in the previous theorem is weakly amenable.*

After reviewing basic facts about relative hyperbolicity, essentially following the treatment of Groves and Manning [GM08] we introduce thinnings in section 3. This is the technical heart of the paper. The final two sections are devoted to the applications: in section 4 we discuss proper affine actions on L^p-spaces, and in section 5 we discuss weak amenability. Some time after our results were announced, Chatterji and Dahmani proved a permanence result for proper affine actions on Banach spaces more general than our application in section 4 [CD18].

Acknowledgement. The authors are grateful for the hospitality extended by their home institutions during visits of the third author to the University of Hawai‘i and of the first author to the Université Paris Sud.

2. Relative hyperbolicity

We shall work in the setting of locally finite graphs. To establish notation, let Γ be a (simplicial) graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$; we shall denote these simply by V and E when no confusion can arise. A graph is *locally finite* if each vertex belongs to only finitely many edges; if there is a uniform bound on the number of edges to which a vertex belongs the graph has *bounded geometry*. We shall equip a graph (or rather, its vertex set) with the *edge path distance* in which the distance between two vertices is the smallest number of edges on an edge path connecting them. A *geodesic* is an edge path which realizes the distance between its endpoints.

Gromov introduced a notion of hyperbolicity for general metric spaces [Gro87]. There are many equivalent definitions, and in our context a convenient choice is the definition in terms of *thin triangles*: a geodesic triangle in a graph Γ is *δ-thin* if each side of the triangle is contained in the δ-neighborhood of the union of the other two sides; and the graph Γ is *hyperbolic* if there exists $\delta > 0$ such that every geodesic triangle in Γ is δ-thin.

Let now G be a finitely generated discrete group, with fixed (finite, symmetric) generating set S. The *Cayley graph* of G (with respect to the generators S) is the graph Γ with vertex set G and in which vertices represented by group elements g and h span an edge precisely when $g^{-1}h$ belongs to S. The group G is *hyperbolic* if its Cayley graph is hyperbolic; since hyperbolicity (for graphs) is a quasi-isometry invariant this is well-defined independent of the choice of generating set.

In this paper, our interest is in *relative hyperbolicity*, also introduced by Gromov. The basic setting here is that of a finitely generated group G together with a finitely generated subgroup P of G. The subgroup P is the *peripheral subgroup*. (It is possible to consider a finite collection of peripheral subgroups, but for simplicity we shall restrict attention to the case of a single peripheral.) Very roughly speaking, G is relatively hyperbolic with respect to P if the geometry of G *between or transverse to* the cosets of P is hyperbolic.

As with hyperbolicity, the precise definition of relative hyperbolicity admits a great many variants; the original definition of Gromov [Gro87] has been reinterpreted by Farb, Bowditch and Osin, for example [Far98, Bow12, Osi06]. We shall work with a characterization given by Groves and Manning [GM08]. To formulate the definition, we fix finite, symmetric generating sets S_1 of P and S of G such that $S_1 \subset S$ and such that $S \smallsetminus S_1$ does not contain any elements of P. Denote the Cayley graph of G with respect to S by Γ. The force of the above setup is that the Cayley graph of P with respect to S_1 appears as the full subgraph of Γ on the subset $P \subset V(\Gamma)$. Similarly, the full subgraph on each (left) coset t of P is isomorphic to the Cayley graph of P; (left) multiplication by any element of the coset gives an isomorphism. Then, G is hyperbolic relative to P if the *cusped space* obtained from Γ by attaching *combinatorial horoballs* to these copies of the Cayley graph of P is hyperbolic in the usual sense. The balance of the section is dedicated to the precise definitions.

Combinatorial horoballs Let Γ be a (typically infinite) graph with vertex set V and edge set E. The *combinatorial horoball* over Γ, denoted $B(\Gamma)$ or simply B when no confusion can arise, is the graph with:

(1) vertex set $V(B) = V \times \mathbb{N}$, and
(2) edge set $E(B)$ with two kinds of edges:
    (a) vertical edges: $(v, n) \sim (v, n+1)$ for all $v \in V$ and $n \in \mathbb{N}$
    (b) horizontal edges: $(v, n) \sim (w, n)$ if $d_\Gamma(v, w) \leq 2^n$.

Vertices of the form (v, n), together with the edges as in (b), comprise *level* n; this is the full subgraph of B on the vertices of the form (v, n). With this terminology, note that the 0^{th}-level is a copy of the original graph Γ and the remaining levels are copies of Γ with extra edges. For a drawing of a piece of the combinatorial horoball over (the Cayley graph of) $\mathbb{Z}$ see figure 1 below.

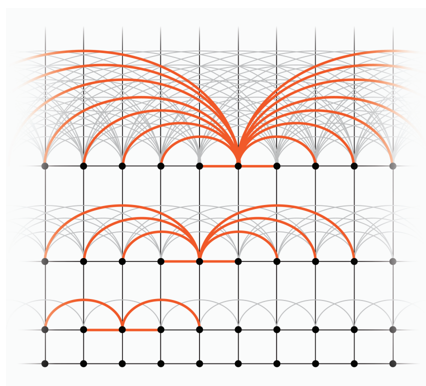

Figure 1: The combinatorial horoball over $\mathbb{Z}$.

Clearly, if Γ is locally finite so is the combinatorial horoball B. Further, if Γ is infinite and has bounded geometry, B is again locally finite but does not have bounded geometry; the valence of vertices increases with their level.

Metric balls in a bounded geometry graph have (at most) exponential growth. While not true for general locally finite graphs, we shall require the following simple result concerning the growth of metric balls in a combinatorial horoball. For the statement, denote the (closed) r-ball with center v by $N_r(v)$; we shall employ this notation consistently throughout and, when confusion could arise, shall indicate with a superscript which graph is under consideration. Recall that a graph has *polynomial growth* if $\#N_r(v) \leq Kr^k$, for some $k \in \mathbb{N}$ and $K > 0$, independent of the center v.

2.1 Proposition. *If Γ as polynomial growth, then the combinatorial horoball B has* *exponential growth for balls centered on level* 0*: there exist* $k \in \mathbb{N}$ *and* $C > 0$ *such that for every vertex* $(v, 0)$ *on level* 0 *we have* $\# N_r^B(v, 0) \le C k^r$*.*

Proof. We have that $N_r^B(v,0) \subset N_{2^r}^{\Gamma}(v) \times \{0, 1, \ldots, r\}$, which has cardinality at most $\# N_{2^r}^{\Gamma}(v) \cdot (r+1) \le C(r+1)(2^r)^k \le C'(2^k+1)^r$. □

<u>The Cusped Space</u> We return now to our group G and subgroup P, with fixed (finite, symmetric) generating sets S and S_1 as above. We then have the Cayley graph $\Gamma = \Gamma(G)$ and, as remarked earlier, the full subgraph of Γ on the vertices in a coset t of P is isomorphic to the Cayley graph of P. The *cusped space* is defined by attaching a combinatorial horoball $B(t)$ to Γ over the coset t. Explicitly, the cusped space is the graph X with

(1) vertex set $G \times \mathbb{N}$, and
(2) edge set with two kinds of edges: horizontal edges
   (a) $(g, 0) \sim (h, 0)$ if $g^{-1}h \in S$, and
   (b) $(g, n) \sim (h, n)$ if $g^{-1}h$ is the product of (at most) 2^n generators in S_1;
   (c) and vertical edges $(g, n) \sim (g, n+1)$ for all $n \in \mathbb{N}$.

2.2 Remark. The full subgraph of X on the vertices at level 0 is the Cayley graph Γ of G. On levels $n > 0$, if $(g, n) \sim (h, n)$ then g and h are in the same coset of P.

Here then is our working definition of relative hyperbolicity [GM08]. As was the case with hyperbolicity, this is independent of the choice of generating sets.

2.3 Definition (Groves-Manning)**.** A finitely generated group G is hyperbolic relative to a finitely generated subgroup P if the associated cusped space X is hyperbolic.

3. Thinning the cusped space

While is natural to exploit the hyperbolicity of the cusped space (or other associated hyperbolic spaces) when studying relatively hyperbolic groups, the lack of bounded geometry can cause difficulties. In this section we introduce a *thinning* technique which enables us, in the classical case in which the peripherals have polynomial growth, to treat the cusped space as though it had bounded geometry. While we hope the thinning technique will be useful elsewhere, we shall apply it in the following sections to the problem of existence of proper actions and to weak amenability.

The problem that we are facing is that we can discretize the space to have bounded geometry, but cannot preserve the action of the group at the same time. The solution will be to organize the space of thinnings into a compact space.

Thinning a horoball Let Γ be a locally finite graph and $B = B(\Gamma)$ its combinatorial horoball. We shall be working with subgraphs of B, each of which has its own graph distance; recall that we denote the closed r-ball *in* B with center v by by $N^B_r(v)$, and similarly for subgraphs.

3.1 Definition. Fix $\alpha \in (0,1]$ and $d, C \in \mathbb{N}$. A subset T of the vertex set of B is an (α, d, C)-*thinning* of B if the following conditions hold:

(1) for every $x \in T$ we have $\#(N^B_1(x) \cap T) \leq d + 1$;
(2) for every $y \in B$ we have $N^B_C(y) \cap T \neq \varnothing$; and
(3) for every $x \in T$ and $r \in \mathbb{N}$, we have $T \cap N^B_{\alpha r}(x) \subset N^T_r(x)$.

By convention a thinning includes all vertices on level 0. When the constants are clear from context, we say simply that T is a *thinning* of B. We denote the (possibly empty) set of thinnings of B by $\mathcal{T}(B)$.

Formally, a thinning T is a subset of the vertex set of B. To interpret metric notions, we regard it also as a full subgraph of B so that it has its own edge path distance (which is not the subspace distance inherited from B). In this way, the closed r-ball $N^T_r(x)$ in condition (3) makes sense. Moreover, all three conditions in the defintion can be interpreted metrically.

3.2 Remark. The above conditions on a thinning imply:

(1) T has bounded geometry, with the valence of each vertex bounded by d;
(2) T is C-coarsely dense in B; and
(3) the inclusion $T \hookrightarrow B$ is a quasi-isometric embedding, and in fact

$$\alpha d_T(x,y) \leq d_B(x,y) \leq d_T(x,y), \qquad \forall x, y \in T.$$

The first issue we need to address is the *existence* of thinnings. The rough idea of the proof of Proposition 3.3 is already clear in the following simple case. In figure 2 below we show a piece of a thinning of (the Cayley graph of) $\mathbb{Z}$. The blue vertices belong to the thinning; these include every vertex on level 0, every second on level 1, every fourth on level 2, etc. Together with the blue edges the thinning is a full subgraph of the combinatorial horoball. In the general case, the vertices of the thinning T will come from a decreasing sequence of nets in Γ. For the statement recall that a graph has *strict polynomial growth* if there exist constants $D \in \mathbb{N}$ and $K \geq 1$, such that

$$K^{-1} r^D \leq \# N^\Gamma_r(w) \leq K r^D,$$

for every $r \geq 1$, independent of the center w. We call D the *degree of growth* of Γ.

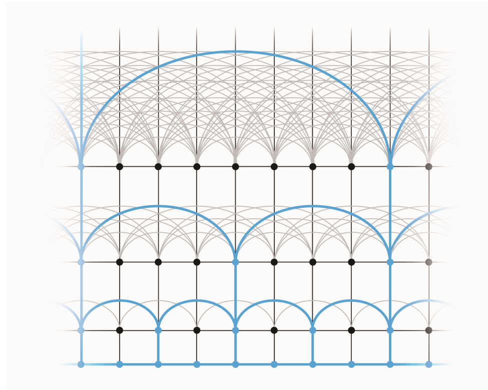

Figure 2: Thinning the combinatorial horoball over $\mathbb{Z}$.

3.3 Proposition. *Let Γ be an (infinite) graph of strict polynomial growth and let B be the combinatorial horoball over Γ. There exists $d_0 \in \mathbb{N}$, depending on the growth function of Γ, such that the space of (α, d, C)-thinnings of B is non-empty, for every $\alpha \le 1/195$, every $d \ge d_0$ and every $C \ge 1$.*

Proof. Let Γ be as in the statement. The general statement follows immediately once we construct a $(1/195, d_0, 1)$-thinning, for some d_0 depending on the growth of Γ. We shall construct our thinning from a decreasing sequence of subsets of Γ. Define these as follows: let $\Gamma_0 = \Gamma$; and for every $n \ge 1$, let Γ_n be a maximal 2^n-separated subset of Γ_{n-1}. Our thinning T is now defined as follows:

$$T = \bigsqcup \Gamma_{n-5} \times \{n\}.$$

In other words, the set Γ_{n-5} comprises the vertex set of T at level n. (By convention $\Gamma_n = \Gamma$ when n is negative.) As a full subgraph of B, vertices of T at level n are connected by a (horizontal) edge when their distance in Γ is at most 2^n.

Before verifying that T satisfies the three conditions in Definition 3.1 we record a *coarse density* property of the Γ_n that we require. By maximality, every Γ_{n-1}-ball of radius 2^n contains at least one element of Γ_n. In other words, Γ_n is 2^n-coarsely dense in Γ_{n-1}:

$$\Gamma_{n-1} \subset \bigcup_{x \in \Gamma_n} N^\Gamma(x, 2^n),$$

where, for this proof only, we write $N^\Gamma(x, 2^n)$ for the ball in Γ of radius 2^n and center x. A simple induction shows then that Γ_n is $6 \cdot 2^n$-coarsely dense Γ:

$$(\dagger) \qquad \Gamma \subset \bigcup_{x \in \Gamma_n} N^\Gamma(x, 6 \cdot 2^n).$$

We can now check that T satisfies the conditions Definition 3.1. The second condition (with $C = 1$) is immediate from $(\dagger)$, which shows that every $z \in B$ at level n is joined by an edge in B to an element $x \in T$ (at the same level); indeed any $x \in \Gamma_{n-5}$ at distance at most $6 \cdot 2^{n-5} \le 2^n$ from z works.

The first condition, regarding the valence, follows from a doubling property of Γ. Fix a vertex $z \in T$ at level n. If another vertex $x \in T$ at level n is connected by an edge to z then the distance in Γ between x and z is at most 2^n and we have

$$N^\Gamma(x, 2^{n-6}) \subset N^\Gamma(z, 2^n + 2^{n-6}) \subset N^\Gamma(z, 2^{n+1}).$$

On the other hand, if x and $y \in T$ are distinct vertices at level n then, by separatedness, their distance in Γ is at least 2^{n-5} so that

$$N^\Gamma(x, 2^{n-6}) \cap N^\Gamma(y, 2^{n-6}) = \varnothing.$$

Put together we see that the 'horizontal valence' of z is not more than the maximum number of disjoint Γ-balls of radius 2^{n-6} a Γ-ball of radius 2^{n+1} can contain. But this is easy to bound using strict polynomial growth. Indeed, suppose D is the degree of growth of Γ. Counting points, we see that a Γ-ball of radius 2^{n+1} cannot contain more than $128^D K^2$ disjoint balls of radius 2^{n-6}. Taking into account the vertical edges, the valence of T is at most $d_0 = 128^D K^2 + 2$.

Finally, we turn to the third condition. We must show that for vertices x, $y \in T$ we have that

$$d_T(x, y) \le 195 \cdot d_B(x, y).$$

For this, let $k = d_B(x, y)$ and consider a geodesic (of length k) in B connecting x and y. By the above, every vertex of this geodesic is at distance at most 1 in B from a vertex of T. Replacing them with these new vertices we obtain a sequence of vertices in T,

$$x = t_0, \ldots, t_k = y,$$

with the property that the distance in B between any two consecutive vertices is at most 3. It therefore suffices to show the following: if two vertices t and $t' \in T$ are such that $d_B(t, t') \le 3$ then $d_T(t, t') \le 195$. Permuting t and t' if necessary, we may assume that t is a vertex on level n, while t' is on level n' and $n' - 3 \le n \le n'$. Since Γ_i is a decreasing sequence, every vertical edge in B from a vertex of T towards a lower level is contained in T. It follows that t' is at distance at most 3 in T from a vertex $s \in T$ at level n, and we are therefore reduced to showing the following: if vertices s and $t \in T$ are both on level n and $d_B(s, t) \le 6$ then $d_T(s, t) \le 192$. Writing $s = (g, n)$ and $t = (h, n)$ for g and $h \in \Gamma$, it follows easily that $d_\Gamma(g, h) \le 6 \cdot 2^{n+3} = 192 \cdot 2^{n-2}$. Consider a sequence of vertices in Γ,

$$g = g_0, \ldots, g_q = h,$$

such that $d_\Gamma(g_i, g_{i+1}) \le 2^{n-2}$, and $q \le 192$. By construction, g_0 and g_q belong to Γ_{n-5}. Using (†) again, choose for each of the remaining g_i a $t_i \in \Gamma_{n-5}$ at a distance at most $6 \cdot 2^{n-5}$. From the triangle inequality we see

$$d_\Gamma(t_i, t_{i+1}) \le 6 \cdot 2^{n-5} + 2^{n-2} + 6 \cdot 2^{n-5} = 20 \cdot 2^{n-5} \le 2^n,$$

so that every two consecutive t_i are joined by an edge in T. The resulting path $t_0, \ldots, t_q$ shows that $d_T(s,t) \leq 192$ and we are done. $\square$

Henceforth, by an *appropriate choice of constants* we shall mean a choice of $\alpha > 0$ and d, $C \in \mathbb{N}$ for which the space $\mathcal{T}(B)$ of (α, d, C)-thinnings is non-empty. We shall topologize the (non-empty) set $\mathcal{T}(B)$ as a subspace of the collection of all subsets of (the vertex set of) B. A convenient description of this topology is that a basic open neighborhood of a subset $Z \subset B$ is

$$(*) \qquad U_F(Z) = \{\, Y \subset B : Y \cap F = Z \cap F \,\},$$

where F is a finite subset of B.

3.4 Proposition. *For each appropriate choice of constants the* (*non-empty*) *set* $\mathcal{T}(B)$ *of* (α, d, C)*-thinnings of* B *is compact.*

Proof. The space of *all subsets* of B is compact in the above topology; indeed, upon identification of the power set of B with the product $\Pi\{\,0,1\,\}$ (indexed over B) in the obvious way, the topology above is the infinite product topology. So, it suffices to see that $\mathcal{T}(B)$ is a closed subset of this space. We shall check that the complement of each condition in Definition 3.1 is open.

For condition (1), let T be a subset of B and suppose there exists an $x \in T$ such that $\#(N_1^B(x) \cap T) > d+1$; then the same holds for every subset belonging to the open neighborhood of T determined by the finite subset $N_1^B(x) \subset B$ according to $(*)$. For condition (2) we suppose instead that there exists a $y \in B$ such that $N_C^B(y) \cap T = \varnothing$; then the same is true for every subset belonging to the open neighborhood of T determined by the finite subset $N_C^B(y) \subset B$. And finally for condition (3) we suppose there exists an $x, y \in T$ and $r \in \mathbb{N}$ such that $y \in N_{\alpha r}^B(x) \subset N_r^B(x)$, but $y \notin N_r^T(x)$; then the same is true for every element of the open neighborhood of T determined by the finite subset $N_r^B(x) \subset B$. $\square$

Suppose now that P is a group of polynomial growth, that Γ is the Cayley graph of P with respect to a fixed (finite, symmetric) generating set, and that $B = B(\Gamma)$ is the combinatorial horoball. It follows from Gromov's polynomial growth theorem [Gro81] and earlier work of Bass [Bas72] that P automatically has strict polynomial growth (there is no elementary proof of this available). Thus we have access to Proposition 3.3 and the space $\mathcal{T}(B)$ of (α, d, C)-thinnings is non-empty, for appropriate choice of constants. In this setting P acts on B by graph automorphisms, with $p \in P$ sending (q,n) to (pq,n). We obtain a continuous action on the space of subsets of B which preserves $\mathcal{T}(B)$. Since P has polynomial growth it is amenable, and $\mathcal{T}(B)$ admits a P-invariant measure.

3.5 Proposition. *For each appropriate choice of constants, the (non-empty) compact space $\mathcal{T}(B)$ of (α, d, C)-thinnings of B admits a P-invariant probability measure. We shall denote one such measure by ν.* □

<u>Thinning the cusped space</u> We return to our original setting: G is a finitely generated group, relatively hyperbolic with respect to a finitely generated subgroup P of polynomial growth; $S_1 \subset S$ are (finite, symmetric) generating sets of P and G, respectively, as above. Recall that the associated cusped space X was constructed by attaching to the Cayley graph of G a combinatorial horoball $B(t)$ over each coset t of P. Building on the construction of $\mathcal{T}(B)$ in the previous section, we shall construct a space $\mathcal{T}(X)$ of *thinnings of the cusped space* X, and equip it with a G-action and G-invariant probability measure. Essentially, a *thinning* of X is obtained by replacing the horoballs attached over the cosets of P by thinned horoballs. Formally then, we define

$$\mathcal{T}(X) = \prod_{t \in G/P} \mathcal{T}(B(t)).$$

We equip $\mathcal{T}(X)$ with the infinite product topology, in which it is compact (and non-empty for each appropriate choice of constants).

As for the probability measure, we should like to use the infinite product of the measure ν of the previous section with itself. Denote by B the combinatorial horoball over the Cayley graph of P, and recall that ν is a P-invariant probability measure on $\mathcal{T}(B)$. For each coset $t \in G/P$ we identify $B \cong B(t)$ and $\mathcal{T}(B) \cong \mathcal{T}(B(t))$ using multiplication by an element $g \in t$ and we consider the push-forward measure $\nu_t = g \cdot \nu$. This is independent of the choice of $g \in t$: if $g_1 \in t$ is another choice we have

$$g_1 \cdot \nu = g(g^{-1} g_1) \cdot \nu = g \cdot \nu,$$

where we use that ν is P-invariant and that $g^{-1} g_1 \in P$. Now the infinite product $\otimes \nu_t$ is a probability measure on $\mathcal{T}(X)$, which we denote μ.

Here then is the first result we require on the space of thinnings of the cusped space.

3.6 Proposition. *The Borel measure μ on $\mathcal{T}(X)$ is G-invariant.*

To understand the statement, G acts on $\mathcal{T}(X)$ through its action on X: an element $T \in \mathcal{T}(X)$ is a family of thinnings $T_t \in \mathcal{T}(B(t))$ of the cosets t; in particular the T_t are a family of (disjoint) subsets of X; multiplication by an element $g \in G$ yields another such family.

For the proof, we shall describe the action more concretely. Selecting coset representatives provides us with 'coordinates' on $\mathcal{T}(X)$. Precisely, if $g_t \in t$ are coset representatives we have

$$(\ddagger) \qquad \prod_{t \in G/P} \mathcal{T}(B) \to \mathcal{T}(X) = \prod_{t \in G/P} \mathcal{T}(B(t)), \qquad (T_t)_t \mapsto (g_t T_t)_t,$$

where now the thinning $T_t \in \mathcal{T}(B)$ is translated by g_t to a thinning $g_t T_t \in \mathcal{T}(B(t))$. Written in these coordinates the action of an element $g \in G$ is given by the associated permutation of the cosets, followed with rotation by elements of P within the individual cosets. Precisely, if s and t are cosets and $g \cdot s = t$ then the composition

$$P \to s \to t \to P$$

is simply multiplication by $p(g,s) = g_t^{-1} g g_s \in P$; here the first and third maps identify the cosets s and t with P by multiplication with the appropriate coset representative (or its inverse), and the middle map is multiplication by g. In terms of the coordinates (‡) then, g acts on the infinite product on the left according to

$$(g \cdot T)_t = p(g,s) T_s, \quad s = g^{-1} \cdot t, \tag{††}$$

in other words as multiplication by $p(g,s) \in P$ from the factor corresponding to s to that corresponding to t.

Proof. Given the discussion above this is essentially obvious: μ is precisely the push forward of the infinite product measure $\otimes\nu$ under the coordinate map (‡) and we have described the action in terms of these coordinates. The measure $\otimes\nu$ is determined by its values on the cylinder sets:

$$\otimes\nu(U) = \prod_{t \in G/P} \nu(U_t), \qquad U = \prod_{t \in G/P} U_t\,;$$

here each U_t is an open set in $\mathcal{T}(B)$ and for all but finitely many t we have $U_t = \mathcal{T}(B)$, so that all but finitely many terms in the product on the left are $= 1$. It suffices to see that the measure of such a U is preserved by the action of an element g of G. But, according to (††) $g \cdot U$ is obtained by permuting the index set and following within each coordinate by an element of P, so that

$$\otimes\nu(g \cdot U) = \otimes\nu\left(\prod_{t \in G/P} p(g, g^{-1} \cdot t) U_{g^{-1} \cdot t}\right) = \prod_{t \in G/P} \nu\left(p(g, g^{-1} \cdot t) U_{g^{-1} \cdot t}\right) = \otimes\nu(U),$$

where we use that ν is P-invariant. □

A thinning $T \in \mathcal{T}(X)$ is a family $T_t \in \mathcal{T}(B(t))$ of thinnings of the combinatorial horoballs over the the individual cosets $t \in G/P$, and in particular a disjoint family of subsets of (the vertex set of) X. The union of these is a subset of X. As before, we consider the full subgraph on this subset and equip it with its graph distance. Abusing notation, we denote the resulting graph and metric space also by T. We require one further result, which relates the geometry of a thinning of the cusped space X to the geometry of X itself.

3.7 Proposition. *Fix an appropriate choice of constants $\alpha \in (0,1]$ and $d, C \in \mathbb{N}$. For every $T \in \mathcal{T}(X)$ the inclusion $T \subset X$ is a quasi-isometry, with quasi-isometry constants independent of T and depending only on the constants. Consequently, these T are δ-hyperbolic for a $\delta > 0$ independent of T.*

Proof. Let $T \in \mathcal{T}(X)$ be a thinned cusped space. First, T is C-coarsely dense in X since each thinned horoball T_t is C-coarsely dense in the corresponding $B(t)$. Next, since T is a subgraph of X, we have for every $x, y \in T$

$$d_X(x,y) \leq d_T(x,y),$$

and it remains only to show that $\alpha d_T(x,y) \leq d_X(x,y)$.

To prove this let x, $y \in T$, and let ω be a geodesic path from x to y in X. As a path in X, the geodesic ω will pass through a number of horoballs. To transit between horoballs ω must to return to level 0 of X because at higher levels the horoballs are disjoint from one another. Hence we may realize ω as the concatenation of subpaths ω_i such that each (except possibly the first and last) begins and ends in level 0 of X, and may be entirely contained in level 0. In particular, each ω_i is either entirely within a single horoball or is a path in the Cayley graph of G.

Suppose that ω_i lies entirely within the horoball $B(t)$ sitting over the coset t. Its endpoints $\omega_{i\pm}$ belong to T_t and we obtain a geodesic path $\widetilde{\omega}_i$ in T_t with the same endpoints. It follows that

$$|\omega_i| = d_{B(t)}(\omega_{i-}, \omega_{i+}) \geq \alpha d_{T_t}(\omega_{i-}, \omega_{i+}) = \alpha |\widetilde{\omega}_i|,$$

where we recall that for each thinned horoball T_t and every $x, y \in T_t$ we have

$$\alpha d_{T_t}(x,y) \leq d_{B(t)}(x,y).$$

See Remark 3.2. In the event that ω_i lies entirely within level 0 of X let $\widetilde{\omega}_i = \omega_i$, which is a geodesic path in X and also in T. Concatenating the $\widetilde{\omega}_i$ we obtain a path $\widetilde{\omega}$ in T from x to y. Conclude that

$$d_X(x,y) = |\omega| = \sum |\omega_i| \geq \sum \alpha |\tilde{\omega}_i| = \alpha |\tilde{\omega}| \geq \alpha d_T(x,y),$$

where we recall that $\alpha \leq 1$.

We have shown that the inclusion $T \subset X$ is a quasi-isometry, with quasi-isometry constants depending only on α and C. By the quasi-isometry invariance of hyperbolicity for geodesic spaces, the T are all δ-hyperbolic for a common δ [BH99, Thm. III.1.9]. □

3.8 Remark. Viewing a thinning T as a subset of X provides an alternate, and quite convenient description of $\mathcal{T}(X)$, its topology and G-action. The space of thinnings

$\mathcal{T}(X)$ homeomorphic to a (closed) subspace of the (compact) space of subsets of X, topologized as in $(*)$. The natural action of G on X induces an action of G by homeomorphisms on the set of subsets of X, which restricts to the (continuous) action of G on $\mathcal{T}(X)$.

4. Proper affine actions

This section contains our first application: the existence of a proper affine action of G on a Banach space. The strategy is to use the thinning technique introduced above to adapt Yu's proof, which is based on an averaging technique of Mineyev, that a hyperbolic group admits a proper affine action on an ℓ^p-space for p sufficiently large [Yu05, Min01]. Later, in an unpublished manuscript, Lafforgue gave a self-contained treatment which incorporates both Mineyev's averaging technique and Yu's proof; we shall follow this unified treatment, which recently appeared in [AL17]. The core technical results of this approach are sumarized in the following proposition; we refer to [AL17] for the proof and relevant notation (see especially [AL17, Thm. 4.1]).

4.1 Proposition. *Let Z be a δ-hyperbolic graph with bounded geometry. There exists a function $\tau : Z \times Z \to \mathrm{Prob}(Z)$ with the following properties:*

(1) *if $\tau(x,a)(b) \neq 0$ and $d_Z(x,a) \geq \Delta$ then $b \in [x,a]_{2\delta}$ and $d_Z(a,b) = \Delta$;*
if $\tau(x,a)(b) \neq 0$ then $b \in [x,a]_{2\delta}$ and $d_Z(a,b) \leq \Delta$; here $\Delta = 4\delta$

(2) *there exits $\varepsilon > 0$ such that $\forall k$, $\exists C_k$ so that*

$$d(x,x') \leq k \Rightarrow \|\tau(x,a) - \tau(x',a)\|_1 \leq C_k e^{-\varepsilon d(x,a)}.$$

Furthermore, if $g : Z \to Z'$ is an isomorphism of δ-hyperbolic graphs we have

(3) *$\tau(x,a)(b) = \tau'(gx, ga)(gb)$.*

The constants depend only on the hyperbolicity constant δ and the bounded geometry of Z (so not on the particular Z). □

In what follows we shall refer to (1) as the *support condition*, to (2) as the *decay condition* and to (3) as the *equivariance condition*. The intuition behind the proposition is clear: $\tau(x,a) \in \ell^1(X)$ is a probability measure supported near a which *indicates the direction from a to x*. See [NY12] for the very relevant and motivating case of the fundamental group of a closed, negatively curved manifold for more.

4.2 Remark. Below, when generalizing to the relatively hyperbolic setting, we shall require analogs of the three conditions in this proposition. The support condition is more involved than the other two, and shall require us to make an observation on the *construction* in the paper [AL17] of Alvarez and Lafforgue that lies behind this proposition. While they took $\Delta = 4\delta$, inspection of the argument there reveals that we may choose any $\Delta \geq 4\delta$ and the proposition still holds, although the other constants in the conclusion may depend on the choice.

Proposition 4.1 has the following routine consequence; the proof is standard, and is included in a form to which we can conveniently refer later.

4.3 Theorem. *Let Z be a δ-hyperbolic graph of bounded geometry, and suppose G acts properly on Z by graph automorphisms. Then G admits a proper affine action on $\ell^p(Z, \ell^1(Z))$, for sufficiently large p.*

Proof. For every $1 \leq p \leq \infty$ we have a linear, isometric representation of G on the Banach space $\ell^p(X, \ell^1(X))$; viewing elements ϕ of this space as two-variable functions the representation is given by the formula

$$g \cdot \phi(a, b) = \phi(g^{-1}a, g^{-1}b).$$

We shall define a formal cocycle for this representation. Let τ be as in the previous proposition, fix $x_0 \in Z$ and define $\phi : Z \to \ell^1(Z)$ by $\phi(a) = \tau(x_0, a)$. Observe that ϕ belongs to $\ell^\infty(Z, \ell^1(Z))$ and so does the formal cocycle $b(g) = g \cdot \phi - \phi$. It remains to check that for sufficiently large p the cocycle belongs to $\ell^p(Z, \ell^1(Z))$, and is proper. It is for this that we shall use the hypotheses of the proposition.

A simple calculation using the equivariance condition gives

$$\begin{aligned} b(g)(a, b) &= \phi(g^{-1}a, g^{-1}b) - \phi(a)(b) \\ &= \tau(x_0, g^{-1}a)(g^{-1}b) - \tau(x_0, a)(b) \\ &= \tau(gx_0, a)(b) - \tau(x_0, a)(b), \end{aligned}$$

so that also

$$(**) \qquad \|b(g)\|^p = \sum_{a \in Z} \|b(g)(a)\|_1^p = \sum_{a \in Z} \|\tau(gx_0, a) - \tau(x_0, a)\|_1^p.$$

This equality is the basis of our analysis of the cocycle.

<u>Well-definedness:</u> The well-definedness of the cocycle follows from the decay condition, combined with the bounded geometry hypothesis which we shall use in the form that metric balls in Z grow at most exponentially. Formally, let d be a uniform bound on the valence of the vertices of Z and let p be large enough so that $e^{-\varepsilon p}d < 1$, where ε is as in the Proposition 4.1. The expression in $(**)$ is then bounded above by

$$\sum_{a \in Z} C e^{-\varepsilon p d(x_0, a)} \leq C \sum_{k=0}^{\infty} e^{-\varepsilon p k} d^k \leq C \sum_{k=0}^{\infty} (e^{-\varepsilon p} d)^k < \infty,$$

where C depends on $d(gx_0, x_0)$ as in Proposition 4.1. So $b(g) \in \ell^p(X, \ell^1(X))$.

Properness: The properness follows from the support condition. To understand this, recall from [AL17] that $b \in [x,a]_{2\delta}$ means that $d(x,b) + d(b,a) \le d(x,a) + 2\delta$. The support condition now easily implies that if a belongs to a geodesic from x_0 to gx_0 and is sufficiently far (at distance at least $\Delta = 4\delta$) from the endpoints then the functions $\tau(x_0,a)$ and $\tau(gx_0,a)$ are disjointly supported. Choose a's evenly spaced out along a geodesic from x_0 to gx_0 so that their number is proportional to the distance between x_0 and gx_0. The norm of each difference $\tau(gx_0,a) - \tau(x_0,a)$ appearing in $(**)$ is then 2, so that this expression is bounded below by a constant proportional to $d(x_0, gx_0)$. Since the action of G on Z is assumed proper, we are through. □

4.4 Remark. We record several remarks on the proof which we shall require later. A careful reading of well-definedness argument above reveals that, beyond the decay condition, we only used the exponential growth of balls in Z centered at the fixed base point x_0. As for the properness, the support condition gives that $\tau(x_0,a)$ is supported near the geodesic from a to x_0 at a prescribed distance from a, which in turn guaranteed disjointness of the supports of $\tau(x_0,a)$ and $\tau(gx_0,a)$. Finally, a free action on a locally finite graph is metrically proper.

Here then is the main result of this section. Our strategy for the proof shall be to mimic the proof of Theorem 4.3 using a suitably modified version of Proposition 4.1.

4.5 Theorem. *Let G be a finitely generated group, relatively hyperbolic with respect to a finitely generated subgroup P of polynomial growth. For sufficiently large p we have:*

1. *G admits a proper action on a mixed ℓ^p-ℓ^1-space; and*
2. *G admits a proper action on a L^p-space.*

Let G and P be as in the statement. Recall our setup: X is the corresponding cusped space for suitable choices of generators; and, for appropriate choice of constants, $\mathcal{T}(X)$ is the space of thinnings of X on which G acts with a G-invariant probability measure μ. We view thinnings $T \in \mathcal{T}(X)$ as full subgraphs of X and as such these have bounded geometry, are hyperbolic and quasi-isometric to X, with valence, hyperbolicity and quasi-isometry constants independent of T.

According to Proposition 4.1, for each $T \in \mathcal{T}(X)$ we have a function

$$\overline{\tau}^T : T \times T \to \mathrm{Prob}(T), \qquad \overline{\tau}^T(x,a)(b) \in [0,1]$$

of three variables, x, a and $b \in T$. We restrict the first variable to G, which appears as level 0 of each thinning T; and we use the extension by 0 to view $\mathrm{Prob}(T) \subset \mathrm{Prob}(X)$ and thereby extend the third variable to all of X. Finally we define

$$\tau^T : G \times X \to \mathrm{Prob}(X), \qquad \tau^T(x,a) = \underset{T \cap N_C^X(a)}{\mathrm{average}}\ \overline{\tau}^T(x,a')$$

as the average over those $a' \in T$ for which $d_X(a, a') \le C$; this is now a function of three variables $x \in G$ and a, $b \in X$. A final average removes the dependence on the thinning:

$$\tau : G \times X \to \mathrm{Prob}(X), \qquad \tau(x, a) = \int_{\mathcal{T}(X)} \tau^T(x, a)\, d\mu(T),$$

also a function of three variables $x \in G$ and a, $b \in X$. We interpret the integral pointwise: fix $a, b \in X$ and $x \in G$ and then $\tau(x, a)(b)$ is the integral of a $[0, 1]$-valued function of $T \in \mathcal{T}(X)$; given the description of the topology on $\mathcal{T}(X)$ in $(*)$ and looking at the construction in [AL17] it is straightforward to check that this function is continuous.

4.6 Remark. The support of $\tau(x, a)$ is finite, and also contained in an X-ball of uniform radius. This follows directly from the first of the following two observations:

(1) if $\tau^T(x, a)(b) \ne 0$ then $d_X(a, b) \le C + \Delta$;
(2) the support of $\tau^T(x, a)$ has cardinality at most N_0, independent of x, a and T.

The metric bound in (1) is straightforward. If $\tau^T(x, a)(b) \ne 0$ then necessarily $b \in T$ and there exists an $a' \in T$ such that both $d_X(a, a') \le C$ and $d_T(a', b) \le \Delta$. It follows that

$$d_X(a, b) \le d_X(a, a') + d_X(a', b) \le d_X(a, a') + d_T(a', b) \le C + \Delta.$$

The cardinality bound in (2) is only slightly more involved. First, observe that the average defining $\tau^T(x, a)$ is over a *uniformly* finite set. If $a' \in T \cap N^X_C(a)$ then for any other $a'' \in T \cap N^X_C(a)$ we have that $\alpha d_T(a', a'') \le d_X(a', a'') \le 2C$ so that the average is over a subset of $N^T_{2C\alpha^{-1}}(a')$. By the uniform bounded geometry condition on thinnings, the cardinality of this set is bounded independent of T.

Next, for every $a' \in T$ the support of $\overline{\tau}^T(x, a')$ is contained in the T-ball of radius Δ and center a'. Again by the uniform bounded geometry condition on thinnings, the cardinality of this set is bounded independent of T. It now follows that the support of $\tau^T(x, a)$ is contained in the finite union of finite sets, with all cardinalities bounded independent of T (and x and a as well). □

As indicated, the proof of Theorem 4.5 shall, using the τ defined above as input, follow the proof of Theorem 4.3. We shall need analogs of the support, decay and equivariance conditions for the τ^T, and for τ itself. We begin with the equivariance condition, the proof of which is immediate from the corresponding statement in Proposition 4.1.

4.7 Lemma (Equivariance)**.** *Let $g \in G$. We have $\tau^{gT}(gx, ga)(gb) = \tau^T(x, a)(b)$, for every $x \in G$, every a and $b \in X$ and every thinning $T \in \mathcal{T}(X)$.* □

4.8 Lemma (Decay)**.** *There exists $\varepsilon > 0$ with the following property: for every k there exists C_k such that for every $x, x' \in G$ and every $T \in \mathcal{T}(X)$ we have*

$$d_X(x,x') \le k \Rightarrow \|\tau^T(x,a) - \tau^T(x',a)\|_1 \le C_k e^{-\varepsilon d_X(x,a)},$$

for every $a \in X$.

Proof. This follows readily from Proposition 4.1. Take ε as in that proposition. Given k the proposition provides a constant $C_{\alpha^{-1}k}$. If now $d_X(x,x') \le k$ then $d_T(x,x') \le \alpha^{-1}k$ for every thinning T so that

$$\|\overline{\tau}^T(x,a') - \overline{\tau}^T(x',a')\|_1 \le C_{\alpha^{-1}k} e^{-\varepsilon d_T(x,a')}$$

for every $a' \in T$. The definition of $\tau^T(x,a)$, and of $\tau^T(x',a)$ involves an average over those $a' \in T$ for which $d_X(a,a') \le C$; for such a' we have

$$d_T(x,a') \ge d_X(x,a') \ge d_X(x,a) - C.$$

Putting things together we get

$$\|\tau^T(x,a) - \tau^T(x',a)\|_1 \le C_{\alpha^{-1}k}\, e^{\varepsilon C}\, e^{-\varepsilon d_X(x,a)}$$

as required. □

4.9 Lemma (Support)**.** *Let $x, y \in G$, and let a belong to a geodesic in X connecting x and y sufficiently far from the endpoints. For every pair of thinnings $T_1, T_2 \in \mathcal{T}(X)$ the supports of $\tau^{T_1}(x,a)$ and $\tau^{T_2}(y,a)$ are disjoint.*

Proof. It is here that we shall use the added flexibility of choosing a large Δ; see Remark 4.2. Let $x \in G$ and $a \in X$ be given; let ω be an X-geodesic connecting a and x. Roughly, we shall show that the support of $\tau^T(x,a)$ is clustered near the X-geodesic ω and at a sufficient distance from a. Precisely, we shall show that if $\tau^T(x,a)(b) \ne 0$ then there exists a b' on ω such that

$$d_X(b,b') \le 4\delta + R \quad \text{and} \quad d_X(a,b') \ge \alpha\Delta - (C + R + 4\delta)$$

for all sufficiently large Δ and a sufficiently far (depending on Δ) from x; here $R = R(\alpha,\delta)$ is a constant depending only on α and δ. Importantly, the inequalities above are independent of the particular thinning T.

From this, the lemma follows easily (and in direct analogy with the support argument given in the proof of Theorem 4.3). Indeed, in the notation of the statement if both $\tau^{T_1}(x,a)(b)$ and $\tau^{T_2}(y,a)(b)$ are non-zero we have b'_x and b'_y on the geodesic connecting x and y at distance at least $2(\alpha\Delta - (C + R + 4\delta))$; on the other hand

$$d_X(b'_x, b'_y) \le d_X(b, b'_x) + d_X(b, b'_y) \le 8\delta + 2R.$$

This is a contradiction for Δ chosen larger than $\alpha^{-1}(8\delta + 2R + C)$.

So, choose such a Δ. Return to $x \in G$, $a \in X$ and an X-geodesic ω connecting them; assume further that $d_X(a, x) \geq \Delta + C$. If $\tau^T(x, a)(b) \neq 0$ then there exists an $a' \in T$ such that $d_X(a, a') \leq C$ and $\overline{\tau}^T(x, a)(b) \neq 0$. We have

$$d_T(a', x) \geq d_X(a', x) \geq d_X(a, x) - d_X(a, a') \geq \Delta + C - C = \Delta,$$

so that from Proposition 4.1 we have $d_T(a', b) = \Delta$ and $b \in [a', x]_{2\delta}$. Choose $b_1 \in T$ on a T-geodesic σ^T connecting a' to x such that $d_T(a', b_1) = \Delta$. An elementary argument then gives that $d_T(b, b_1) \leq 3\delta$; see Lemma 3.3 in [AL17].

Let now σ be an X-geodesic connecting the same points a' and x. Being a T-geodesic, σ^T is an X-quasi-geodesic by Proposition 3.7 (in fact, it is α^{-1}-bi-Lipschitz in X). So the X-Hausdorff distance between σ^T and σ is at most R, for some constant R depending only on δ and α^{-1} [BH99, Thm. III.1.7]. It follows that there exists $b_2 \in \sigma$ such that $d_X(b_1, b_2) \leq R$ which gives

$$d_X(b, b_2) \leq d_X(b, b_1) + d_X(b_1, b_2) \leq d_T(b, b_1) + d_X(b_1, b_2) \leq 3\delta + R.$$

Finally, consider a geodesic triangle in X with corners a, a', x, and with the X-geodesic ω forming the side from a to x, and σ forming the side from a' to x. This triangle is δ-slim so that for the point b_2 on σ there is a point b' belonging either to ω or to the side between a and a', satisfying $d_X(b_2, b') \leq \delta$. For such a point it follows immediately that

$$d_X(b, b') \leq d_X(b, b_2) + d_X(b_2, b') \leq 3\delta + R + \delta = 4\delta + R,$$

as required. It remains to show that b' lies on ω and to bound its X-distance to a below.

We claim b' cannot be on the side between a and a' and so much be on the X-geodesic ω. For the bound, begin by observing that

$$d_X(a', b_2) \leq d_X(a', a) + d_X(a, b') + d_X(b', b_2) \leq C + d_X(a, b') + \delta,$$

so that also

$$\begin{aligned} d_X(a, b') &\geq d_X(a', b_2) - (C + \delta) \geq d_X(a', b) - d_X(b, b_2) - (C + \delta) \\ &\geq d_X(a', b) - (C + R + 4\delta) \geq \alpha d_T(a', b) - (C + R + 4\delta) \\ &= \alpha\Delta - (C + R + 4\delta), \end{aligned}$$

as required. The check that b' belongs to ω is similar: since $d_X(a, a') \leq C$ a point on σ that is within δ of a point on the side a to a' has X-distance at most $\delta + C$ from a', but we know that $d_X(a', b_2)$ is at least $\alpha\Delta - (3\delta + R) \geq 5\delta + R + C$, which would be a contradiction. □

Proof of Theorem 4.5. The proof of the first statement is completely analogous to the proof of Theorem 4.3. Above we have defined a function $\tau : G \times X \to \mathrm{Prob}(X)$ and, following the proof of Theorem 4.3, we define

$$b(g) = g \cdot \phi - \phi, \qquad \phi \in \ell^\infty(X, \ell^1(X)), \quad \phi(a) = \tau(x_0, a)$$

for some fixed $x_0 \in G$ (for example, x_0 = the identity of G). We shall show that $b(g) \in \ell^p(X, \ell^1(X))$ for sufficiently large p, and for this we must only verify appropriate analogs of the hypotheses of Proposition 4.1 for our function τ and make a few additional remarks.

The equivariance follows immediately from Lemma 4.7 and the G-invariance of the measure μ on $\mathcal{T}(X)$. As for the well-definedness, the decay condition follows from Lemma 4.8 upon averaging over $T \in \mathcal{T}(X)$. Also, as described in Remark 4.4 we need exponential growth of metric X-balls with center in G, which follows from Proposition 2.1; while that proposition was stated only for a single combinatorial horoball the generalization to the cusped space is immediate using finite generation of G. And finally, given our work above in Lemma 4.9 one checks easily that the supports of $\tau(x, a)$ and $\tau(y, a)$ are disjoint for a on a geodesic in X connecting x and y that is sufficiently far from both x and y.

Let us turn to the second statement. We shall replace the space $\ell^p(X, \ell^1(X))$ on which G is currently acting by an L^p-space in two steps. For the first step, we modify the function τ so that it takes values in $\ell^2(X)$ instead of $\mathrm{Prob}(X) \subset \ell^1(X)$. We do this using the *Banach-Mazur map*, defined for a unit vector $u = (u_x) \in \ell^1(X)$ by

$$\beta(u)_x = |u_x|^{1/2} \operatorname{sign}(u_x);$$

β is a homeomorphism from the unit sphere in $\ell^1(X)$ onto the unit sphere of $\ell^2(X)$ and it satisfies the inequalities

$$\frac{1}{2}\|u - v\|_1 \leq \|\beta(u) - \beta(v)\|_2 \leq \sqrt{2}\|u - v\|_1^{1/2},$$

for $u, v \in \ell^1(X)$ of norm 1. See [BL00, Chapter 9.1]. Further, β preserves supports and is G-equivariant when each sphere is equipped with the norm-preserving action of G coming from its action on the set X. With these observations, the proof above carries through immediately to give a proper cocycle for the action of G on $\ell^p(X, \ell^2(X))$, for all sufficiently large p.

In the second step, we shall replace $\ell^2(X)$ by the space $L^p(\Omega, \mu)$ for some probability space (Ω, μ) equipped with a measure preserving action of G; the unitary action of G on $\ell^p(X, L^p(\Omega, \mu))$ will be the natural one and we must show this representation admits a proper cocycle. The key to this is a standard construction that converts an orthogonal representation of a (locally compact) group G into a subrepresentation of a representation coming from a measure preserving action on some standard Borel probability space, the details of which are presented in [BdlHV08, Appendix A7]. (See also [NP08] for a related argument.) We shall apply this construction to the representation of G on $\ell^2(X)$. We conclude, combining [BdlHV08, Theorem A.7.13] with [BdlHV08, Example A.7.6], that there exists a standard Borel probability space (Ω, μ) on which G acts by measure preserving transformations such that $\ell^2(X)$ is isomorphic as a representation of G to a subrepresentation of $L^2(\Omega, \mu)$ corresponding to a Gaussian Hilbert subspace $\mathcal{K}$. Recall that a *Gaussian Hilbert subspace* is a Hilbert subspace $\mathcal{K} \subset L^2(\Omega, \mu)$ with the property that every $X \in \mathcal{K}$ is a centered Gaussian random variable. Putting everything together, we have a proper cocycle b for the natural representation of G on $\ell^p(X, L^2(\Omega, \mu))$, and the values of b belong to the subspace $\ell^p(X, \mathcal{K})$.

Now we claim that the cocycle b is a proper cocycle for the natural representation of G on $\ell^p(X, L^p(\Omega, \mu))$ as well. For this observe that the Gaussian Hilbert space $\mathcal{K}$ is in fact contained in each $L^p(\Omega, \mu)$ for finite p, and moreover that the L^p-norm on $\mathcal{K}$ is simply the L^2-norm multiplied by the L^p-norm of the standard Gaussian random variable (which is finite). It follows easily that the values of b belong to $\ell^p(X, L^p(\Omega, \mu))$ and that viewed in this way b remains proper; it is a cocycle on formal grounds alone. □

4.10 Remark. In the previous proof it is tempting to omit the second step of the argument and move directly from $\ell^1(X)$ to $\ell^p(X)$ using the appropriate Banach-Mazur map. This is problematic because the Hölder constant of the Banach-Mazur map map depends on p (it is only $1/p$-Hölder).

5. Weak amenability

This section contains our second application: weak amenability of G. Weak amenability and ideas surrounding it were introduced and developed by Haagerup and various coauthors in a series of papers [CH89, DCH85]. Most of the initial papers focus primarily on locally compact groups and their lattices, and more specifically on rank 1 semi-simple Lie groups. In the context of discrete groups weak amenability was quickly seen to be equivalent to an approximation property of the (reduced) group C*-algebra [Haa16]. A general reference for this is [BO08].

We recall the definitions. Let S be a set. A kernel $\phi : S \times S \to \mathbb{C}$ is *completely bounded* if there exists a Hilbert space $\mathcal{H}$ and uniformly bounded functions $\alpha, \beta : S \to \mathcal{H}$ such that

$$\phi(x, y) = \langle \alpha(x), \beta(y) \rangle;$$

in this case the *completely bounded norm* (cb-norm) of ϕ is at most the product of the ℓ_∞-norms of α and β. Precisely,

$$\|\phi\|_{cb} = \inf_{\alpha,\beta} \| \alpha \|_\infty \| \beta \|_\infty,$$

where the infimum is taken over all possible maps α and β as above. Let now G be a (countable discrete) group. A function $f : G \to \mathbb{C}$ *completely bounded* if the corresponding kernel $\phi : G \times G \to \mathbb{C}$ defined by $\phi(g,h) = f(h^{-1}g)$ is completely bounded, and in this case $\| f \|_{cb} = \| \phi \|_{cb}$. Finally, the group G is *weakly amenable* (with Cowling-Haagerup constant C) if there exists a sequence of finitely supported functions $\{f_n\} : G \to \mathbb{C}$ such that f_n converges pointwise to 1 and each f_n is completely bounded with cb-norm at most C.

The above discussion raises the possibility of using geometric properties of G, or of a space on which G acts, to construct kernels or functions which are completely bounded; the α and β required to show complete boundedness are constructed geometrically. The first construction of this type was put forward by Bozejko and Picardello who showed that free products of amenable groups are weakly amenable [BP93]. Their construction works directly on the Cayley graph of a free group and gives a very easy proof that finitely generated free groups are weakly amenable. Ozawa later generalized the construction and used it to show that hyperbolic groups are weakly amenable [Oza08], and this is our starting point. Our strategy is, as above, to use the thinning technique to adapt the construction to prove that classical relatively hyperbolic groups are weakly amenable. Here then is the main result of this section, and an immediate corollary which is apparently not known by other means.

5.1 Theorem. *A finitely generated group G which is relatively hyperbolic with respect to a finitely generated subgroup P of polynomial growth is weakly amenable.*

5.2 Corollary. *Let H be a hyperbolic group and P be a group of polynomial growth. The free product $H \star P$ is weakly amenable. In particular, the free product $H \star \mathbb{Z}^n$ is weakly amenable.*

We shall be terse, having given a quite detailed treatment in the case of proper actions in the previous section. The results we need from Ozawa's work on hyperbolic groups are summarized in the following proposition. (See [Oza08], Thm. 1, and its proof.)

5.3 Proposition. *Let Z be a bounded geometry hyperbolic graph. There exists a constant C depending only on the bounded geometry and hyperbolicity constants of Z with the following properties: for every $r \in (0,1)$ and every $n \in \mathbb{N}$*

(1) *the kernel $(x,y) \mapsto r^{d_Z(x,y)}$ is completely bounded with cb-norm at most C;*
(2) *the characteristic function of the set $E_Z(n) = \{ (x,y) : d_Z(x,y) \leq n \}$ is completely bounded with cb-norm at most $C(n+1)$.* □

5.4 Remark. As a formal consequence of the above we have: for every sequence $r_n \nearrow 1$ with $r_n \in (0,1)$ there exists a sequence $R_n \nearrow \infty$ with $R_n \in \mathbb{N}$ such that the kernel

$$\phi_n^Z(x,y) = \begin{cases} r_n^{d_Z(x,y)}, & \text{if } d_Z(x,y) \leq R_n \\ 0, & \text{else} \end{cases}$$

is completely bounded, with cb-norm at most $2C$. It is these kernels that we shall use below. Observe that if Z is the Cayley graph of a hyperbolic group G then the kernels ϕ_n are left invariant and come from functions on G which witness its weak amenability.

We return to the relatively hyperbolic case, and shall use the notation of the previous sections. For each thinning $T \in \mathcal{T}(X)$ we have a sequence of kernels $\phi_n^T : T \times T \to [0,1]$ and, since the T are uniformly δ-hyperbolic and have uniformly bounded geometry, the cb-norms of these kernels are bounded independently of T (and of course also of n): there exists $C < \infty$ such that $\| \phi_n^T \|_{cb} \leq 2C$ for all T. We restrict both variables to G, which appears as level 0 of every thinning and note that this does not increase the cb-norm. And as in the the previous section, we eliminate the dependence on T by integration:

$$\phi_n(x,y) = \int_{\mathcal{T}(X)} \phi_n^T(x,y)\, d\mu(T), \tag{‡‡}$$

for x and $y \in G$. We interpret the integral pointwise, in light of the following lemma.

5.5 Lemma. *Let $x, y \in X$. For every $r \in (0,1)$ and every $n \in \mathbb{N}$ the function*

$$T \longmapsto \begin{cases} r^{d_T(x,y)}, & \text{if } d_T(x,y) \leq n \\ 0, & \text{else} \end{cases}$$

is continuous on $\mathcal{T}(X)$.

Proof. This follows easily from the following assertion: the set of all $T \in \mathcal{T}(X)$ for which $d_T(x,y) \leq n$ (so also $x, y \in T$) is a clopen set. To see that it is open, suppose T is such that $d_T(x,y) \leq n$. If $F \subset X$ is the (finite) set of vertices along a T-geodesic from x to y then every T_1 belonging to the basic open neighborhood of T defined by F contains F and so satisfies $d_{T_1}(x,y) \leq n$. To see that its complement is open, suppose T is such that $d_T(x,y) > n$. If $F \subset X$ is the (finite) set of vertices belonging to a path in X of length $\leq n$ connecting x and y then every T_2 belonging to the basic open neighborhood of T defined by F satisfies $d_{T_2}(x,y) > n$; otherwise the vertices on a path of length $\leq n$ connecting x and y in such a T_2 would belong to $T_2 \cap F$, and hence also to T which is a contradiction. □

Proof of Theorem 5.1. We first show that each ϕ_n is G-invariant. The action of $g \in G$ on X induces a graph isomorphism $T \to gT$, and so also an isometry of these graphs. It follows easily that $\phi_n^{gT}(gx, gy) = \phi_n^T(x,y)$ for every x, $y \in T$ and, in particular for every x, $y \in G$. Applying the G-invariance of the measure μ the result follows:

$$\phi_n(gx,gy) = \int_{\mathcal{T}(X)} \phi_n^T(gx,gy)\, d\mu(T) = \int_{\mathcal{T}(X)} \phi_n^{g^{-1}T}(x,y)\, d\mu(T) = \phi_n(x,y).$$

The left invariant the kernel ϕ_n arises in the usual way from the real-valued function $x \mapsto \phi_n(e,x)$ on G, where e denotes the identity element of G. We must show that this function is finitely supported and converges to 1 pointwise. These assertions follow from properties of the ϕ_n^T and the fact that the X and T-distances are bi-Lipschitz when restricted to G, with constants independent of T.

More precisely, if $d_X(e,x) > R_n$ then also $d_T(e,x) > R_n$ for every T and the integrand in (‡‡) is identically 0. Since the X-distance is proper and left invariant on G finiteness of the support follows. As for the convergence, for sufficiently large n we have that $R_n \geq \alpha^{-1} d_X(e,x) \geq d_T(e,x)$ so that the integrand in (‡‡) satisfies

$$r_n^{\alpha^{-1} d_X(e,x)} \leq \phi_n^T(e,x) \leq 1,$$

independent of T. So, $\phi_n(e,x)$ satisfies similar inequalities and the result follows.

It remains only to estimate the cb-norm of the ϕ_n. This is more involved, and we shall treat it in the following lemma. □

5.6 Lemma. *The ϕ_n are completely bounded, with cb-norm at most $2C$.*

For the proof we record a few details regarding the kernels ϕ_n^Z introduced in Remark 5.4, in particular why they have cb-norm at most $2C$. As remarked, this is a formal consequence of Proposition 5.3. We introduce the kernels

$$\phi_{r,R}^Z(x,y) = E_Z(R)(x,y)\, r^{d_Z(x,y)},$$

and observe that

$$\phi_{r,R}^Z(x,y) = \sum_{k=0}^{R} \left(E_Z(k) - E_Z(k-1)\right) r^k = r^{d_Z(x,y)} - \sum_{k=R+1}^{\infty} \left(E_Z(k) - E_Z(k-1)\right) r^k,$$

where we abuse notation by writing $E_Z(k)$ for the characteristic function of this set, and understand $E_Z(-1) = 0$. Now, given $r_n \nearrow 1$ we select $R_n \nearrow \infty$ such that for each n the tail in the above expression has cb-norm $\leq C$:

$$\sum_{k=R_n+1}^{\infty} \| \, E_Z(k) - E_Z(k-1) \, \|_{cb} \, r_n^k \leq \sum_{k=R_n+1}^{\infty} 2C(k+1)\, r_n^k \leq C,$$

where we have applied Proposition 5.3. Applying the proposition again gives that the cb-norm of $\phi_n^Z = \phi_{r_n,R_n}^Z$ is at most $2C$. The value in this analysis is that it shows quite concretely how the kernels ϕ_n^Z are constructed from the more primitive kernels provided by Proposition 5.3. Each of the algebraic operations involved (essentially addition and subtraction) has an analog at the level of the Hilbert space valued α, β and it is this that we shall exploit below.

Proof. It is clear from the previous discussion that the kernels ϕ_n^T appearing in the integrand (‡‡) are constructed from the primitive kernels

$$(x,y) \mapsto r^{d_T(x,y)} \quad \text{and} \quad (x,y) \mapsto E_T(k)(x,y)$$

in identical fashion, independent of T; this is because the estimates in Proposition 5.3 are independent of T. Suppose $\alpha_i^T, \beta_i^T : G \to \mathcal{H}$ are Hilbert space valued functions realizing the conclusions of Proposition 5.3 in the sense that for every $x, y \in G$ we have

$$r^{d_T(x,y)} = \langle\, \alpha_{1,r}^T(x), \beta_{1,r}^T(y)\,\rangle \quad \text{and} \quad E_T(k)(x,y) = \langle\, \alpha_{2,k}^T(x), \beta_{2,k}^T(y)\,\rangle;$$

and suppose further these are normalized so that, for example, $\|\, \alpha_{1,r}^T(x)\,\| \le \sqrt{C}$ and $\|\, \alpha_{2,k}^T(x)\,\| \le \sqrt{C(k+1)}$ for every $x \in G$, etc. It is then clear that the kernel ϕ_n^T is realized by

$$\alpha_n^T(x) = \alpha_{1,r_n}^T(x) \oplus \bigoplus_{k=R_n+1}^{\infty} r_n^{k/2}\left(\alpha_{2,k}^T(x) \oplus \alpha_{2,k-1}^T(x)\right)$$
$$\beta_n^T(y) = \beta_{1,r_n}^T(y) \oplus \bigoplus_{k=R_n+1}^{\infty} r_n^{k/2}\left(-\beta_{2,k}^T(y) \oplus \beta_{2,k-1}^T(y)\right);$$

these are functions $G \to \overline{\mathcal{H}}$ where $\overline{\mathcal{H}}$ is an appropriate large direct sum of copies of $\mathcal{H}$. Further, the constituent α's and β's give, for each x and $y \in G$ respectively, measurable (in the weak sense) functions $\mathcal{T}(X) \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space constructed from the ℓ^2-space on the (countable) collection of finite subsets of X. This follows from the detailed construction of these in Ozawa's paper. In particular, $\alpha_n^T(x)$ and $\beta_n^T(y)$ are themselves measurable functions of T, for every x and $y \in G$, respectively.

To estimate the cb-norm of the kernels ϕ_n we assemble the above data into functions from G into the direct integral Hilbert space as follows:

$$\alpha_n, \beta_n : G \longrightarrow \int_{\mathcal{T}(X)}^{\oplus} \overline{\mathcal{H}}, \quad \alpha_n(x) = \left(T \mapsto \alpha_n^T(x)\right), \quad \beta_n(y) = \left(T \mapsto \beta_n^T(y)\right),$$

From here, everything is a direct calculation. First, these α_n and β_n represent the kernel ϕ_n:

$$\langle\, \alpha_n(x), \beta_n(y)\,\rangle = \int_{\mathcal{T}(X)} \langle\, \alpha_n^T(x), \beta_n^T(y)\,\rangle \, d\mu(T) \int_{\mathcal{T}(X)} \phi_n^T(x,y)\, d\mu(T) = \phi_n(x,y);$$

and second the norm of $\alpha_n(x)$ is easily seen to be

$$\begin{aligned}
\|\, \alpha_n(x)\,\|^2 &= \int_{\mathcal{T}(X)} \|\, \alpha_n^T(x)\,\|^2 \, d\mu(T) \\
&= \int_{\mathcal{T}(X)} \left(\|\, \alpha_{1,r_n}^T(x)\,\|^2 + \sum_{k=R_n+1}^{\infty} \left(\|\, \alpha_{2,k}^T(x)\,\|^2 + \|\, \alpha_{2,k-1}^T(x)\,\|^2 \right) r_n^k \right) d\mu(T) \\
&\le \int_{\mathcal{T}(X)} \left(C + \sum_{k=R_n+1}^{\infty} 2C(k+1) r_n^k \right) d\mu(T) \le 2C,
\end{aligned}$$

and similarly for $\beta_n(y)$. It follows that the cb-norm of ϕ_n is $\le 2C$, as required. □

References

[AL17] Aurélien Alvarez and Vincent Lafforgue. Actions affines isométriques propres des groupes hyperboliques sur des espaces ℓp. *Expositiones Mathematicae*, 35(1):103–118, March 2017.

[Bas72] H. Bass. The degree of polynomial growth of finitely generated nilpotent groups. *Proc. London Math. Soc. (3)*, 25:603–614, 1972.

[BdlHV08] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's Property (T)*, volume 11 of *New mathematical monographs*. Cambridge University Press, 2008.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BL00] Y. Benyamini and J. Lindenstrauss. *Geometric nonlinear functional anlysis*, volume 48 of *Colloquium Publications*. American Mathematical Society, Providence, R.I., 2000.

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.

[BP93] Marek Bożejko and Massimo A. Picardello. Weakly amenable groups and amalgamated products. *Proc. Amer. Math. Soc.*, 117(4):1039–1046, 1993.

[CD18] Indira Chatterji and François Dahmani. Proper actions on ℓ^p spaces for relatively hyperbolic groups. arXiv:1801.08047, `http://arxiv.org/abs/1801.08047`, 2018.

[CH89] Michael Cowling and Uffe Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.*, 96(3):507–549, 1989.

[DCH85] Jean De Cannière and Uffe Haagerup. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Amer. J. Math.*, 107(2):455–500, 1985.

[Far98] Benson Farb. Relatively hyperbolic groups. *Geometric and Functional Analysis*, 8(5):810–840, 1998.

[GM08] Daniel Groves and Jason Fox Manning. Dehn filling in relatively hyperbolic groups. *Israel Journal of Mathematics*, 168(1):317–429, 2008.

[Gro81] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[Gro87] M. Gromov. Hyperbolic groups. In S. Gersten, editor, *Essays in Group Theory*, volume 8 of *MSRI Publ.*, pages 75–263. Springer Verlag, 1987.

[Haa16] Uffe Haagerup. Group C^*-algebras without the completely bounded approximation property. *J. Lie Theory*, 26(3):861–887, 2016.

[Min01] I. Mineyev. Straightening and bounded cohomology of hyperbolic groups. *Geom. Funct. Anal.*, 11(4):807–839, 2001.

[NP08] Assaf Naor and Yuval Peres. Embeddings of discrete groups and the speed of random walks. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 076, 34, 2008.

[NY12] Piotr W. Nowak and Guoliang Yu. *Large scale geometry*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2012.

[Osi05] D. Osin. Asymptotic dimension of relatively hyperbolic groups. *Int. Math. Res. Not.*, (35):2143–2161, 2005.

[Osi06] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.

[Oza08] Narutaka Ozawa. Weak amenability of hyperbolic groups. *Groups Geom. Dyn.*, 2(2):271–280, 2008.

[Roe05] John Roe. Hyperbolic groups have finite asymptotic dimension. *Proc. Amer. Math. Soc.*, 133(9):2489–2490, 2005.

[Yu05] Guoliang Yu. Hyperbolic groups admit proper affine isometric actions on l^p-spaces. *Geom. Funct. Anal.*, 15(5):1144–1151, 2005.

Department of Mathematics, University of Hawaii at Manoa, Honolulu, HI, USA
E-mail address: `erik@math.hawaii.edu`

Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland
Current: Computer Science Department, University of Colorado, Boulder, CO,

USA
E-mail address: eric.reckwerdt@colorado.edu

Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université de Paris, Paris, France
E-mail address: tessera@phare.normalesup.org