# OPERATOR NORM LOCALIZATION FOR LINEAR GROUPS AND ITS APPLICATION TO K-THEORY

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ABSTRACT. We prove the operator norm localization property for linear groups. As an application we prove the coarse Novikov conjecture for box spaces of a linear group.

## 1. INTRODUCTION

The operator norm localization property allows one to estimate operator norm locally relative to a metric space [CTWY]. The metric sparsification property is a geometric condition sufficient for the operator norm localization property. While both the sparsification and norm localization properties were introduced for the purpose of studying problems in operator K-theory, they seem to be of independent interest.

**Theorem A.** *Let* $G$ *be countable subgroup of* $GL(n, K)$*, for a field* $K$*. Viewed as a metric space equipped with a proper length metric* $G$ *has the metric sparsification property and hence also the operator norm localization property.*

Observe that as the sparsification and norm localization properties are coarse invariants, and any two proper length metrics are coarsely equivalent, the conclusion is in fact independent of the choices of the proper length metric.

The coarse Novikov conjecture for a (bounded geometry) metric space asserts the injectivity of the coarse Baum-Connes assembly map. As an application of the above result, we prove the coarse Novikov conjecture for box spaces associated to a linear group. Recall that a box space $\Box G$ of a countable residually finite group $G$ is the 'well-spaced' union of a separating family of its finite quotients.

**Theorem B.** *Let* $G$ *be a finitely generated subgroup of* $GL(n, K)$ *for a field* $K$*. The coarse Novikov conjecture holds for every box space* $\Box G$.

In general, $\Box G$ is a sequence of expanders – while the counterexamples of Higson-Lafforgue-Skandalis shows that the coarse Baum-Connes assembly map cannot be surjective for such $\Box G$ our result shows it is, in fact, injective in many cases of interest.

The paper is organized as follows. Sections 2 and 3 are devoted to linear groups; in the first of these we collect a few preliminaries and in the second we prove Theorem A. In Section 4 we discuss a variant of the (strong) Novikov conjecture, *relative* to a family of subgroups. The final

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section contains the proof of Theorem B. In the appendix, we give an alternative proof (in the finite index case) of the fact, due to Chabert-Echterhoff [CE], that the Baum-Connes conjecture with coefficients passes to subgroups. Our argument is based on an equivalent formulation of the Baum-Connes conjecture which uses the localization algebra.

## 2. Preliminaries on linear groups

A *norm*[1] on a field $K$ is a map $d : K \to [0, \infty)$ satisfying, for all $x, y \in K$

(a) $d(x) = 0 \Leftrightarrow x = 0$
(b) $d(xy) = d(x)d(y)$
(c) $d(x + y) \leq d(x) + d(y)$

A norm obtained as the restriction of the usual absolute value on $\mathbb{C}$ via a field embedding $K \to \mathbb{C}$ is *archimedean*. A norm satisfying the stronger *ultra-metric inequality*

(d) $d(x + y) \leq \max\{ d(x), d(y) \}$

in place of the triangle inequality (c) is *non-archimedean*. If in addition the range of $d$ on $K^\times$ is a discrete subgroup of the multiplicative group $(0, \infty)$ the norm is *discrete*.

2.1. **Definition.** [GHW] A field $K$ is discretely embeddable if for every finitely generated subring $A$ of $K$ there exists a sequence $(d_n)$ of norms on $K$ with the following property: For every sequence $R_n > 0$, the subset

$$\{a \in A,\ d_n(a) \leq R_n, \forall n \in \mathbb{N}\}$$

is finite.

The main observation in [GHW] is that a finitely generated field is discretely embeddable.

Let $d$ be a norm on a field $K$. Guentner-Higson-Weinberger define a pseudo-length function $\ell_d$ on $GL(m, K)$ as follows: if $d$ is discrete

$$\ell_d(g) = \log \max_{ij} \{ d(g_{ij}), d(g^{ij}) \}, \tag{2.1}$$

where $g_{ij}$ and $g^{ij}$ are the matrix coefficients of $g$ and $g^{-1}$, respectively; if $d$ is archimedian, arising from an embedding $K \hookrightarrow \mathbb{C}$ then

$$\ell_d(g) = \log \max\{ \|g\|, \|g^{-1}\| \}, \tag{2.2}$$

where $\|g\|$ is the norm of $g$ viewed as an element of $GL(m, \mathbb{C})$, and similarly for $g^{-1}$. The following proposition is central to our discussion of linear groups.

2.2. **Proposition.** *Let $d$ be an archimedean or a discrete norm on a field $K$. The group $GL(m, K)$, equipped with the left-invariant pseudo-metric induced by $\ell_d$, has finite asymptotic dimension.*

The result in the archimedean case follows immediately from the corresponding result for $GL(m, \mathbb{C})$; indeed, the metric on $GL(m, K)$ is the subspace metric it inherits from an embedding into $GL(m, \mathbb{C})$. For $GL(m, \mathbb{C})$ the result follows from known results, once we observe that the length function (2.2) is continuous, hence bounded on compact sets, and proper, meaning that bounded sets are compact.

[1] Guentner-Higson-Weinberger use the term valuation.

The discrete case is more subtle than the archimedean case, primarily because we do not assume that $K$ is locally compact. In this case the result was proven by Matsnev [Ma] (see also [GTY] for a shorter proof).

2.3. **Corollary.** *Let $G$ be a finitely generated linear group equipped with a word metric $d$. Then there exists a sequence $(d_n)$ of pseudo-metrics on $G$ such that for every $n$ the metric space $(G, d_n)$ has finite asymptotic dimension, and such that for any sequence $R_n > 0$, there exists $R > 0$ such that*

$$\bigcap_n B_n(R_n) \subset B(R),$$

*where $B_n(R_n) = \{g\colon d_n(1, g) \leq R_n\}$, and $B(R) = \{g\colon d(1, g) \leq R\}$.*

*Proof.* Follows from [GHW, Theorem 2.2] and from the above proposition. □

## 3. METRIC SPARSIFICATION AND OPERATOR NORM LOCALIZATION

Chen-Tessera-Wang-Yu introduced the metric sparsification property to provide a geometric criterion for the operator norm localization property [CTWY].

3.1. **Definition.** A countable (pseudo-)metric space $(X, d)$ has the (metric) sparsification property if for every $\varepsilon > 0$ and $r > 0$ there exists $R > 0$ such that the following statement holds: for every finite measure $\mu$ on $X$ there exists $\Omega \subset X$ with $\mu(\Omega) \geq (1-\varepsilon)\mu(X)$ and a decomposition $\Omega = \bigsqcup \Omega_i$ where

(a) $\operatorname{diam} \Omega_i \leq R$, for every $i$;
(b) $d(\Omega_i, \Omega_j) \geq r$, for every pair $i \neq j$.

Other than the definition, we shall require only a very few facts about the sparsification property. First, sparsification is a coarse property, and as such can be applied unambiguously to countable discrete groups. Second, for such groups, the sparsification property is closed under increasing unions. In particular, a countable group has the sparsification property if all of its finitely generated subgroups do. See [CTWY, Cor 3.6 and Prop. 3.9]. Finally, we require one additional permanence result.

3.2. **Proposition.** *Let $(X, d)$ be a metric space and let $(d_n)$ be a sequence of pseudo-metrics on $X$ satisfying the following:*

(a) *for every $n \in \mathbb{N}$, $d_n \leq d$;*
(b) *for every sequence of positive numbers $(R_n)$, there exists $R > 0$ such that for all $x \in X$,*

$$\bigcap_n B_n(x, R_n) \subset B(x, R),$$

*where $B_n(x, R_n) = \{y \in X\colon d_n(x, y) \leq R_n\}$.*

*If $(X, d_n)$ has the sparsification property for every $n \geq 0$ then $(X, d)$ has the sparsification property as well.*

For the proof, a small reformulation of Definition 3.1 will be convenient. We introduce the relevant terminology. Two elements $x$ and $y$ of a metric space $X$ are $r$-*connected* if there exists a sequence $x_0, x_1, \ldots, x_n$ for which $x_0 = x$, $x_n = y$ and the distance between any two consecutive points is at most $r$. This is an equivalence relation, and the equivalence classes are the $r$-*connected components of* $X$. Now, $X$ has the sparsification property if for every $\varepsilon > 0$ and $r > 0$ there exists $R > 0$ such that the following statement holds: for every finite measure $\mu$ on $X$ there exists $\Omega \subset X$ with $\mu(\Omega) \geq (1-\varepsilon)\mu(X)$ and such that the $r$-connected components of $\Omega$ (in the subspace metric) have diameter at most $R$.

*Proof.* We assume that $(X, d_n)$ has the sparsification property for each $n$; we shall show that $(X, d)$ has the sparsification property as well. Let $\varepsilon > 0$ and $r > 0$ be given. Let $\varepsilon_n > 0$ be a sequence for which

$$\Pi(1 - \varepsilon_n) \geq 1 - \varepsilon.$$

Apply the hypothesis on $(X, d_n)$ to obtain $R_n$ with the following property: for every finite measure $\nu$ on $X$ there exists a subset $\Omega' \subset X$ for which $\nu(\Omega') \geq (1 - \varepsilon_n)\nu(X)$, and for which the $r$-connected components for the metric $d_n$ have $d_n$-diameter at most $R_n$. Apply the hypothesis regarding the $R_n$ to obtain $R$.

We define a sequence of subsets of $X$ as follows. First, $\Omega^{(0)} = X$. Next, assume $\Omega^{(n)}$ has been defined and define $\Omega^{(n+1)}$ by

$$\Omega^{(n+1)} = \Omega^{(n)} \cap \Omega',$$

where $\Omega'$ is the result of applying the hypothesis on $(X, d_n)$ with the measure $\nu(A) = \mu(A \cap \Omega^{(n)})$. In particular, we have

$$\mu(\Omega^{(n+1)}) = \nu(\Omega') \geq (1 - \varepsilon_n)\nu(X) = (1 - \varepsilon_n)\mu(\Omega^{(n)}).$$

The subsets $\Omega^{(n)}$ are nested so that if we define $\Omega = \cap \Omega^{(n)}$ we see that

$$\mu(\Omega) = \lim \mu(\Omega^{(n)}) \geq (\Pi(1 - \varepsilon_n))\, \mu(X) \geq (1 - \varepsilon)\, \mu(X).$$

It remains only to see that the $r$-connected components of $\Omega$ in the metric $d$ have $d$-diameter at most $2R$. Let us denote the $r$-connected component of $x$ by $C_r(x; \Omega, d)$; similar notation will be clear from context. Now, since $d_n \leq d$ we see that

$$C_r(x; \Omega, d) \subset C_r(x; \Omega^{(n)}, d_n) \subset B_n(x, R_n),$$

for every $n$. It follows that $C_r(x; \Omega, d) \subset B(x, R)$ as required. □

3.3. **Theorem.** *A countable linear group has the sparsification property when viewed as metric space with a proper length metric.*

*Proof.* A finitely generated linear group has the sparsification property. Indeed, noting that a countable (pseudo-)metric space of finite asymptotic dimension has the sparsification property [CTWY, Rem. 3.2, Prop. 3.3], we apply Corollary 2.3 and Proposition 3.2. □

Let $X$ be a countable metric space. Fix a separable, infinite dimensional Hilbert space $\mathfrak{H}$. An operator $T$ acting on $\ell^2(X) \otimes \mathfrak{H}$ is represented as a matrix $(T_{x,y})$ with respect to the orthogonal decomposition $\ell^2(X) \otimes \mathfrak{H} = \oplus_{x \in X}(\delta_x \otimes \mathfrak{H})$. The operator $T$ has *finite propagation* if there exists an $r$ such that $T_{x,y} = 0$ when $d(x,y) > r$; in this case, the *propagation* of $T$ is the smallest such $r$.

3.4. **Definition.** A countable metric space $X$ has the (operator) norm localization property if there exists $0 < c \leq 1$ such that for every $r > 0$ there exists $R > 0$ for which the following statement holds: if $T$ has propagation $r$ then there exists a unit vector $\xi \in \ell^2(X) \otimes \mathfrak{H}$ for which

$$\operatorname{diam}(\operatorname{supp}(\xi)) \leq R, \quad \text{and} \quad \|T\xi\| \geq c\|T\|.$$

As was the case with the sparsification property, we shall require very few facts about the norm localization property. First, the norm localization property is a coarse property, and as such can be applied to a countable discrete group. Second, a metric space with the sparsification property also has the norm localization property. See [CTWY, Props. 2.5 and 4.1]. As an immediate consequence we obtain:

3.5. **Theorem.** *A countable linear group has the norm localization property when viewed as a metric space with a proper length metric.* □

## 4. The Novikov conjecture relative to a family of subgroups

Let $G$ be a countable discrete group. The Baum-Connes assembly map for $G$ with coefficients in the $G$-$C^*$-algebra $A$ takes the form

$$KK_*^G(\mathcal{E}G, A) \to K_*(C_r^*(G, A)); \tag{4.1}$$

here $C_r^*(G,A)$ is the reduced crossed product and $\mathcal{E}G$ is the universal proper $G$-space. See [BCH]. The Baum-Connes conjecture (with coefficients) asserts that (4.1) is an isomorphism for every coefficient algebra, whereas the (strong) Novikov conjecture asserts that (4.1) is (split) injective.

Both the Baum-Connes and Novikov conjectures pass from a group to its subgroups. While this result was part of the folklore for some time, a complete account of it has only recently become available [CE]. Precisely, for a subgroup $H$ of $G$ there is an identification of assembly maps

$$\begin{tikzcd} KK_*^G(\mathcal{E}G, c_0(G/H)) \arrow[r] \arrow[d, "\cong"'] & K_*(C_r^*(G, c_0(G/H))) \arrow[d, "\cong"] \\ KK_*^H(\mathcal{E}H, \mathbb{C}) \arrow[r] & K_*(C_r^*(H)). \end{tikzcd}$$

While we have not done so, it is possible to incorporate coefficients in this diagram. Thus, a proof of the Novikov conjecture for a particular group yields at once a proof for each of its finite index subgroups and one may expect these proofs to exhibit a certain uniformity. We shall formalize this idea by introducing the Novikov conjecture *relative to a family of subgroups*. See (4.3) below.

We shall find it convenient to work with a reformulation of (4.1). The *Rips complex of* $G$ *at scale* $d$ is the simplicial complex $P_d(G)$ with vertices the elements of $G$ and in which a subset of vertices spans a simplex if the corresponding elements are pairwise at distance at most $d$. The group $G$, and its subgroups, act simplicially on $P_d(G)$.

While we shall reformulate the domain of assembly in terms of Rips complexes, we shall reformulate the range in terms of Roe algebras. Let $\mathfrak{H}$ be a separable and infinite dimensional Hilbert space. An operator $T$ on $\ell^2(G) \otimes \mathfrak{H}$ is *locally compact* if the operators $T_{x,y}$ appearing in its matrix representation are compact. Equipping $G$ with a proper length metric, the notion of propagation of $T$ is defined. The *Roe algebra of* $G$ is the operator norm closure of $\mathbb{C}[|G|]$, the algebra of all locally compact operators with finite propagation. We shall denote the Roe algebra by $C^*(|G|)$.

Suppose now that $H$ is a subgroup of $G$. An operator $T$ is $H$-invariant if the individual operators $T_{x,y}$ appearing in its matrix representation satisfy $T_{gx,gy} = T_{x,y}$, for every $g \in H$, and $x, y \in G$. The $H$-invariant elements of $\mathbb{C}[|G|]$ form a subalgebra; its norm closure is $C^*(|G|)^H$, the $H$-*equivariant Roe algebra*. Despite the notation, the $H$-equivariant Roe algebra should not be confused with the algebra of $H$-invariant elements of $C^*(|G|)$. For a discussion see [R2].

4.1. **Lemma.** *We have a Morita equivalence* $C^*(|G|)^G \sim C^*_r(G)$. *More generally, for a finite index subgroup we have a Morita equivalence* $C^*(|G|)^H \sim C^*_r(H)$. □

4.2. **Lemma.** *We have an isomorphism* $C^*_r(G, \ell^\infty(G, \mathcal{K})) \cong C^*(|G|)$. *More generally, for a subgroup* $H$ *with quotient* $\Gamma = G/H$ *we have an isomorphism* $C^*_r(G, \ell^\infty(\Gamma, \mathcal{K})) \cong C^*(|G|)^H$. □

After these preliminaries, we can reformulate the assembly map (4.1) for a finite index subgroup $H$ of $G$ in the following convenient form:

$$\mu^H : \lim_{d\to\infty} K^H_*(P_d(G)) \to K_*(C^*(|G|)^H). \tag{4.2}$$

We pause briefly to relate the two versions (4.1) and (4.2) of assembly. The space of finitely supported probability measures on $G$ provides, for every finite index subgroup $H$ of $G$, a model for $\mathcal{E}H$. This model contains the individual $P_d(G)$ as $H$-invariant and $H$-compact subsets and indeed, these are cofinal in the collection of all such subsets. This identifies the domains of (4.1) and (4.2). For a detailed treatment of the more substantive issue of identifying the assembly maps themselves we refer to [R2].

4.3. **Definition.** Let $\{ G_i \}$ be a family of finite index subgroups of a countable group $G$. The uniform product $\Pi^u C^*(|G|)^{G_i}$ is the closure, in the norm coming from the Roe algebra, of the collection of sequences of elements of uniformly bounded norm and uniformly finite propagation. Precisely, it is the norm completion of the following algebra:

$$\{ (a_1, a_2, \dots) : a_i \in \mathbb{C}(|G|)^{G_i} \text{ and } \sup_i \|a_i\| < \infty, \sup_i \operatorname{prop}(a_i) < \infty \}.$$

4.4. **Definition.** Let $\{ G_i \}$ be a family of finite index subgroups of a countable group $G$. The Novikov conjecture for $G$, relative to the family $\{ G_i \}$, is the assertion that the assembly map

$$\lim_{d\to\infty} \prod K^{G_i}_*(P_d(G)) \to K_* \left( \prod\nolimits^u C^*(|G|)^{G_i} \right) \tag{4.3}$$

is injective.

4.5. **Remark.** The intuition is clear. Up to an approximation, projections in the uniform product have controlled propagation, and the same is true of the invertibles implementing the equivalences in the definition of $K_0$. The Novikov conjecture relative to the collection $\{ G_i \}$ asserts, in particular, that when elements $x_i \in K_0^{G_i}(P_d(G))$ satisfy $\mu^i(x_i) = 0$ in $K_0(C^*(|G|)^{G_i})$ uniformly then $x_i = 0$ in $K_0^{G_i}(P_{d'}(G))$, for some $d' \geq d$ independent of $i$.

4.6. **Theorem.** *Let $\{ G_i \}$ be a family of finite index subgroups of a countable group $G$ and let $\Gamma_i = G/G_i$. If the Baum-Connes assembly map for $G$ with coefficients in $\Pi_i \ell^\infty(\Gamma_i, \mathcal{K})$ is injective then $G$ satisfies the Novikov conjecture relative to $\{ G_i \}$.*

*Proof of Theorem 4.6.* The coefficient algebra is

$$\Pi_i \ell^\infty(\Gamma_i, \mathcal{K})) = \{ (\phi_1, \phi_2, \dots) \colon \phi_i \in \ell^\infty(\Gamma_i, \mathcal{K}) \text{ and } \sup_i \|\phi_i\| < \infty \},$$

with the evident norm and componentwise $G$-action. The essential point is that we have an isomorphism

$$C_r^*(G, \Pi_i \ell^\infty(\Gamma_i, \mathcal{K}))) \cong \Pi^u \left( C^*(|G|)^{G_i} \right).$$

Indeed, keeping in mind the requirement of uniform propagation in the definition of the uniform product this follows directly from Lemma 4.2. This takes care of the right hand side of assembly. As for the left hand side,

$$KK_*^G(P_d(G), \Pi_i \ell^\infty(\Gamma_i, \mathcal{K}))) \cong \Pi_i KK_*^G(P_d(G), \ell^\infty(\Gamma_i, \mathcal{K})) \cong \Pi_i K_*^{G_i}(P_d(G)),$$

where we employ a standard 'induction' isomorphism. It remains only to identify the assembly maps themselves – here we refer to [CE, R2]. We provide in the appendix an independent proof based on the localization algebra. □

We close with two results, which follow immediately from Theorem 4.6 – in each case the hypothesis implies the Novikov conjecture for an arbitrary choice of coefficients. See [Y2] and [GHW].

4.7. **Theorem.** *A countable discrete group that is coarsely embeddable in Hilbert space satisfies the Novikov conjecture relative to every family of finite index subgroups.* □

4.8. **Theorem.** *A countable linear group satisfies the Novikov conjecture relative to every family of finite index subgroups.* □

## 5. Application to the coarse Novikov conjecture

In this section we shall prove the coarse Novikov conjecture for a box space associated to a finitely generated linear group. The coarse assembly map for a bounded geometry metric space $X$ takes the form

$$\lim_{d \to \infty} K_*(P_d(X)) \to K_*(C^*(X)); \tag{5.1}$$

here $P_d(X)$ is the *Rips complex* of $X$ as scale $d$ – defined exactly as in the case of a group, $P_d(X)$ is the simplicial complex with vertex set $X$ and in which a finite subset of vertices span a simplex precisely when they are pairwise at distance at most $d$; $C^*(X)$ is the *Roe algebra* of $X$ – the $C^*$-subalgebra of bounded operators on $\ell^2(X) \otimes \ell^2$ generated by the locally compact operators having finite propagation, where $\ell^2$ is a separable and infinite dimensional Hilbert space.

The *coarse Baum-Connes conjecture* for $X$ asserts that (5.1) is an isomorphism and *the coarse Novikov conjecture* for $X$ asserts that (5.1) is injective. In our statements we have restricted attention to spaces of *bounded geometry*, meaning that for every $R > 0$ there exists $N > 0$ such that every $R$-ball contains at most $N$ elements. While it is possible to state the conjectures in greater generality, the coarse Novikov conjecture is false in the absence of the bounded geometry hypothesis – for a counterexample see [Y2]. Roe's book [R1] serves as a basic reference for the coarse Baum-Connes conjecture and relevant definitions.

We are interested in the coarse Novikov conjecture for a 'box space' associated to a countable residually finite group. Let $G$ be such a group. Let $\{ G_i \}$ be a collection of finite index subgroups of $G$ satisfying $G_1 \supset G_2 \supset \ldots$ and $\cap \tilde{G}_i = \{ 1 \}$, where $\tilde{G}_i$ denotes the normal subgroup generated by $G_i$. Such a family is *separating*. Denote the quotients by $\Gamma_i = G/G_i$ and let $\square G = \bigsqcup_{i=1}^{\infty} \Gamma_i$ be the (disjoint) union of the $\Gamma_i$.

We view $\square G$ as a metric space as follows. Equip $G$ with a proper length metric, and equip each $\Gamma_i$ with the quotient metric:

$$d(\overline{x}, \overline{y}) = \min\{ d(a, b) : a \in \overline{x},\, b \in \overline{y} \}, \tag{5.2}$$

where we have introduced the notation $\overline{x}$ for the coset $xG_i$. As the $\Gamma_i$ are finite metric spaces, it is not difficult to see that there exists a metric on $\square G$ satisfying the following two properties: first, its restriction to each $\Gamma_i$ is the quotient metric (5.2) and second, it is *well-spaced* in the sense that

$$\lim_{i+j \to \infty,\, i \neq j} d(\Gamma_i, \Gamma_j) = \infty. \tag{5.3}$$

Equipped with such a metric $\square G$ is the *box space* (of $G$ relative to the collection $\{ G_i \}$). It is independent, up to coarse equivalence, of the choice of metric on $G$ and of the choice of well-spaced metric; indeed, for different choices the identity map will be a coarse equivalence.

5.1. **Remark.** Let us shortly explain why we used the term *separating*, and where this assumption will be used in the sequel. Equip $G$ with a word metric. Now, if $(G_i)$ is separating there exists, for every $d > 0$, an $i_0$ such that $B(1, d) \cap \tilde{G}_i = \{1\}$ for all $i \geq i_0$. But since $\tilde{G}_i$ is normal, this implies that for any $x \in G$, and $g \in \tilde{G}_i \setminus \{1\}$, with $i \geq i_0$, we have $d(x, gx) > d$. In particular, the action of $\tilde{G}_i$ (thus of $G_i$) on $P_d(G)$ is such that each simplex is moved to a disjoint simplex. As a consequence, $P_d(G)/G_i$ identifies as a $G_i$-space with $P_d(\Gamma_i)$. This fact will be used in the proof of Theorem 5.2 below.

5.2. **Theorem.** *Let $G$ be a countable residually finite group, and let $\{ G_i \}$ be a separating family of finite index subgroups of $G$. Assume $G$ has the operator norm localization property. The strong Novikov conjecture for $G$ relative to $\{ G_i \}$ implies the coarse Novikov conjecture for $\square G$.*

Our theorem implies the coarse Novikov conjecture for many interesting examples of expander sequences. Indeed, if $G$ is an infinite property $T$ group, or more generally if $G$ has property $\tau$ with respect to the family $\{ G_i \}$, then $\square G$ is (contains) a sequence of expanders. Examples of such $G$ satisfying the hypothesis of the theorem abound; indeed, many satisfy the hypothesis of the next theorem which, in light of the results of Sections 2 and 4, is an immediate consequence.

5.3. **Theorem.** *The coarse Novikov conjecture holds for every box space associated to a countable, residually finite linear group. In particular, the coarse Novikov conjecture holds for every box space associated to a finitely generated linear group.* □

Analogous results at the level of maximal Roe $C^*$-algebra are proved in [GWY] – they do not require the norm localization property but do require the additional hypothesis that $G$ admits a finite classifying space $BG$. A basic tool in their proof is a lifting construction for the maximal Roe algebra. We begin by discussing a version of lifting for the (reduced) Roe algebra.

Recall the construction of Roe algebra. Fix a separable and infinite dimensional Hilbert space $\mathfrak{H}$. With respect to the Hilbert space decomposition $\ell^2(X) \otimes \mathfrak{H} = \oplus_{x \in \square G}(\delta_x \otimes \mathfrak{H})$ a (bounded) operator $T$ acting on $\ell^2(\square G) \otimes \mathfrak{H}$ is represented by a matrix:

$$T = (T_{x,y})_{x,y \in \square G}.$$

An operator $T$ has finite propagation when $T_{x,y} = 0$ for every pair of sufficiently distant $x$ and $y$; it is locally compact when $T_{x,y}$ is compact for every $x$ and $y$. We denote by $\mathbb{C}[\square G]$ the algebra of all locally compact operators of finite propagation. The Roe algebra $C^*(\square G)$ is the operator norm closure of $\mathbb{C}[\square G]$.

Let now $T \in \mathbb{C}[\square G]$ and suppose $T$ has finite propagation $l$. For each $n$ we may decompose $\square G$ as a disjoint union

$$\square G = \left( \bigsqcup_{i=1}^{n} \Gamma_i \right) \sqcup \bigsqcup_{i=n+1}^{\infty} \Gamma_i.$$

For sufficiently large $n$ the operator $T$ will respect this decomposition, meaning that it will decompose as a block diagonal sum

$$T = T^0 \oplus \prod_{i=n+1}^{\infty} T^i,$$

where $T^0$ acts on $\ell^2(\sqcup_1^n \Gamma_i) \otimes \mathfrak{H}$ and for $i \neq 0$ each $T^i$ acts on $\ell^2(\Gamma_i) \otimes \mathfrak{H}$ – indeed, denoting the ball of radius $2l$ and center the identity element of $G$ by $B(1, 2l)$, any $n$ for which both $B(1, 2l) \cap G_n = \{ 1 \}$ and $d(\Gamma_i, \Gamma_j) > 2l$ for distinct $i, j \geq n$ will suffice.

We shall lift $T$ by lifting the individual $T^i$. We define the lift $S^i$ of $T^i$ to be the operator acting on $\ell^2(G) \otimes \mathfrak{H}$ with $(x, y)$-matrix entry

$$S^i_{x,y} = \begin{cases} T^i_{\overline{x},\overline{y}}, & \text{if } d(x,y) \leq l \\ 0, & \text{otherwise,} \end{cases}$$

where $T^i_{\overline{x},\overline{y}}$ is the $(\overline{x}, \overline{y})$-matrix entry of $T^i$. Define a map:

$$\phi(T) = 0 \oplus \prod_{i=n+1}^{\infty} S^i, \qquad \phi : \mathbb{C}[\square G] \to \prod_{i=1}^{\infty} \mathbb{C}[|G|]^{G_i} \Big/ \bigoplus_{i=1}^{\infty} \mathbb{C}[|G|]^{G_i}.$$

It is not difficult to verify that $\phi$ is an (algebraic) $*$-homomorphism. The lifting result we require is summarized:

5.4. **Proposition.** *Assume* $G$ *has operator the norm localization property. The map* $\phi$ *extends to* (*bounded*) $*$-*homomorphism*

$$\phi : C^*(\Box G) \to \prod^{u} C^*(|G|)^{G_i} \Big/ \bigoplus C^*(|G|)^{G_i}.$$

*Proof.* Immediate from the definition of the operator norm localization property. □

*Proof of Theorem 5.2.* In the course of the proof, we shall encounter diagrams in which some of the groups and morphisms depend on the parameter $d$, and shall need to consider direct limits as $d \to \infty$. We express injectivity of a direct limit morphism by saying an arrow is *injective at infinity*. We similarly employ the phrase *isomorphism at infinity*.

Let $G$ and $\{ G_i \}$ be as in the statement. Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus K_*^{G_i}(P_d(G)) \arrow[r] \arrow[d] & \prod K_*^{G_i}(P_d(G)) \arrow[r] \arrow[d] & \frac{\prod K_*^{G_i}(P_d(G))}{\oplus K_*^{G_i}(P_d(G))} \arrow[r] \arrow[d, "\overline{\mu}"] & 0 \\
0 \arrow[r] & \bigoplus K_*(C^*(|G|)^{G_i}) \arrow[r] & K_*\left(\prod^u C^*(|G|)^{G_i}\right) \arrow[r] & K_*\left(\frac{\prod^u C^*(|G|)^{G_i}}{\bigoplus C^*(|G|)^{G_i}}\right) \arrow[r] & 0.
\end{tikzcd}$$

The first vertical arrow is the sum of Baum-Connes assembly maps for the subgroups $G_i$ and, as the subgroups $G_i$ all satisfy the Novikov conjecture, it is injective at infinity – it is important here that we are dealing with the *sum* and not the *product*. The second vertical arrow is the assembly map for $G$ relative to the family $\{ G_i \}$ and is injective at infinity by hypothesis.

The third vertical map requires some explanation. First, the bottom sequence is a portion of the degenerate 6-term exact sequence for the ideal $\bigoplus C^*(|G|)^{G_i}$ of $\prod^u C^*(|G|)^{G_i}$. To see that the 6-term sequence indeed degenerates observe that the inclusion of the sum into the product factors through the $K$-theory of the uniform product

$$\bigoplus K_*(C^*(|G|)^{G_i}) \to K_*\left(\prod^u C^*(|G|)^{G_i}\right) \to \prod K_*(C^*(|G|)^{G_i}).$$

As the composition is injective, so is the first map, and the 6-term sequence degenerates. Now, the third vertical map is defined by requiring comutativity in the diagram. Further, it is injective at infinity – as subgroups of $K_*\left(\prod^u C^*(|G|)^{G_i}\right)$ we have

$$\lim_{d\to\infty} \prod K_*^{G_i}(P_d(G)) \cap \bigoplus K_*(C^*(|G|)^{G_i}) = \lim_{d\to\infty} \bigoplus K_*^{G_i}(P_d(G)).$$

The map we have just constructed, $\overline{\mu}$, is actually a 'lift' of the coarse Baum-Connes assembly map for $\Box G$. To make this precise, we fix $d$ and analyze the $K$-homology of $P_d(\Box G)$. For sufficiently large $n$ depending only on $d$ we have the following two properties: first, by Remark 5.1, for every $i \geq n$ the subgroup $G_i$ acts freely (and properly) on $P_d(G)$, moreover $P_d(\Gamma_i)$ identifies as $G_i$-space with the quotient space $P_d(G)/G_i$; second $d(\Gamma_i, \Gamma_j) > d$ provided $i, j \geq n$. Introducing the notation $\square_n = \cup_1^n \Gamma_i$ we see that

$$\begin{aligned} K_*(P_d(\square G)) &\cong K_*(P_d(\square_n)) \oplus \prod\nolimits_n^\infty K_*(P_d(\Gamma_i)) \\ &\cong K_*(P_d(\square_n)) \oplus \prod\nolimits_n^\infty K_*^{G_i}(P_d(G)), \end{aligned}$$

where we further identify the K-homology of the quotient $P_d(\Gamma_i) = P_d(G)/G_i$ with the $G_i$-equivariant K-homology of $P_d(G)$. As a consequence, the top row in the following diagram is defined, and exact:

$$\begin{tikzcd} 0 \arrow[r] & K_*(P_d(\square_n)) \oplus \oplus_{i=n}^\infty K_*(P_d(\Gamma_i)) \arrow[r] \arrow[d] & K_*(P_d(\square G)) \arrow[r] \arrow[d, "\mu"] & \frac{\Pi K_*^{G_i}(P_d(G))}{\oplus K_*^{G_i}(P_d(G))} \arrow[r] \arrow[d, "\overline{\mu}"] & 0 \\ & K_*(C^*(\square_n)) \oplus \oplus_{i=n}^\infty K_*(C^*(\Gamma_i)) \arrow[r] & K_*(C^*(\square G)) \arrow[r, "\phi_*"] & K_*\left(\frac{\prod^u C^*(|G|)^{G_i}}{\oplus C^*(|G|)^{G_i}}\right) & \end{tikzcd}$$

Commutativity of the diagram follows from the definition of assembly – for the right hand square, in particular, we also use compatibility of the isomorphism $K_*^{G_i}(P_d(G)) \cong K_*(P_d(\Gamma_i))$ with assembly and the lifting map $\phi$. For details see [GWY].

Now, the leftmost vertical map is an isomorphism at infinity for the following reason. An element in the sum, as a finite sequence, is supported on summands below some fixed $m$ and, as $d \to \infty$, will eventually be absorbed into the first term. The entry on the lower left depends on $d$ through $n$ and similar reasoning applies to it. Thus, the assertion reduces to the fact that assembly is an isomorphism for a bounded metric space. Similar arguments yield the final piece of the puzzle – the first map on the bottom row is injective at infinity.

Recall that the rightmost vertical map, $\overline{\mu}$ is injective at infinity. A diagram chase shows that $\mu$ is injective at infinity, as required. □

## Appendix A. Localization

The key tool employed in Section 4 involved passing from assembly for a group to assembly for its subgroup by taking appropriate coefficients. Here, we shall give a new treatment, via localization algebras, of the relevant result of Chabert-Echterhoff [CE]. We freely employ the notation introduced previously.

Endow the Rips complex $P_d(G)$ with the *simplicial metric*, the maximal metric whose restriction to each simplex is the metric obtained by identifying that simplex with the standard simplex in Euclidean space. As a matter of convention, the distance between points in different connected components of $P_d(G)$ is infinity.

Fix countable dense G-invariant subsets $X_d$ of the individual $P_d(G)$, which are nested in the sense that $X_d \subset X_{d'}$ when $d \le d'$. Equip each $X_d$ with the metric inherited from $P_d(G)$. Fix a separable and infinite dimensional Hilbert space $\mathfrak{H}$ with a G-action such that, for each $x \in P_d(G)$, $\mathfrak{H}$ is isomorphic to $\ell^2(G_x) \otimes \mathfrak{H}_0$ as $G_x$-Hilbert spaces, where $G_x$ is the finite isotropy subgroup of G at $x$ and $\mathfrak{H}_0$ is a Hilbert space with a trivial $G_x$-action. We remark that such a G-Hilbert space

$\mathfrak{H}$ always exists – we can take $\mathfrak{H}$ to be $\ell^2(G) \otimes \mathfrak{K}$ for some separable and infinite dimensional Hilbert space $\mathfrak{K}$ with a trivial $G$ action. The condition on the G-Hilbert space $\mathfrak{H}$ means that $\mathfrak{H}$ contains all unitary representations of the finite isotropy groups and this condition implies that the G-invariant Roe algebra of $P_d(G)$ is Morita equivalent to the reduced group $C^*$-algebra $C^*_r(G)$ (see Definition A.1 below for the concept of invariant Roe algebra). If $G$ is torsion free, then we can choose the Hilbert space $\mathfrak{H}$ to be a Hilbert space with a trivial G-action. More generally, in Definitions A.1, A.2, A.4 and Theorem A.3 below, if the subgroup $H$ of $G$ acts on $P_d(G)$ freely, then we can choose the Hilbert space $\mathfrak{H}$ to be a Hilbert space with a trivial G-action.

We shall define several algebras of operators on the Hilbert space $\ell^2(X_d) \otimes \mathfrak{H}$. We remark that the definitions are independent of the choices of the countable dense subset $X_d$ of $P_d(G)$.

A.1. **Definition** (Roe algebras)**.** We extend the definition of the Roe algebra to $P_d(G)$ as follows:

(a) The Roe algebra $C^*(P_d(G))$ is the operator norm completion of the algebra $\mathbb{C}[X_d]$ of locally compact operators of finite propagation.
(b) If $H$ is a subgroup of $G$, the H-invariant Roe algebra $C^*(P_d(G))^H$ is the operator norm completion of the the subalgebra $\mathbb{C}[X_d]^H$ of all H-invariant elements in $\mathbb{C}[X_d]$.

A.2. **Definition** (Localization algebras)**.** We define localized versions of the above Roe algebras as follows:

(a) The algebraic localization algebra $C^*_{L,alg}(P_d(G))$ is the algebra of bounded and uniformly continuous functions $f : [0,\infty) \to C^*(P_d(G))$ satisfying $\mathrm{prop}(f(t)) \to 0$ as $t \to \infty$.
(b) The localization algebra $C^*_L(P_d(G))$ is the closure of $C^*_{L,alg}(P_d(G))$ with respect to the norm $\|f\| = \sup_{t\in[0,\infty)}\|f(t)\|$.
(c) If $H$ is a subgroup of $G$, the H-equivariant algebraic localization algebra $C^*_{L,alg}(P_d(G))^H$ is the algebra of all H-invariant elements in $C^*_{L,alg}(P_d(G))$.
(d) If $H$ is a subgroup of $G$, the H-equivariant localization algebra $C^*_L(P_d(G))^H$ is the closure of $C^*_{L,alg}(P_d(G))^H$.

The Roe algebras and localization algebras are related by the *evaluation homomorphism*

$$\mathrm{ev}^H : C^*_L(P_d(G))^H \to C^*(|G|)^H = \lim_{d\to\infty} C^*(P_d(G))^H, \quad \mathrm{ev}^H(f) = f(0).$$

We have a *local assembly map*

$$\mu^H_L : K^H_*(P_d(G)) \to K_*(C^*_L(P_d(G))^H), \tag{A.1}$$

defined in the same manner as the local index map in [Y1]. The local assembly map and the assembly map (4.2) are related by the evaluation homomorphism according to

$$\mu^H = (e^H)_* \circ \mu^H_L. \tag{A.2}$$

The proof of the following result is identical to the proof of the main result in [Y1] is therefore omitted. See also [S] for a detailed treatment of the local assembly map in the equivariant setting when $H$ is torsion free.

A.3. **Theorem.** *The local assembly map $\mu^H_L$ is an isomorphism.* □

It is possible, and useful to incorporate *coefficients* into the above discussion. We shall be deliberately brief. Let $A$ be a G-$C^*$-algebra, faithfully and equivariantly represented on a G-Hilbert space $\mathfrak{K}$. We shall define several algebras of operators acting on the Hilbert space $\ell^2(X_d) \otimes \mathfrak{H} \otimes \mathfrak{K}$, where $\mathfrak{H}$ is a G-Hilbert space such that, for each $x \in P_d(G)$, $\mathfrak{H}$ is isomorphic to $\ell^2(G_x) \otimes \mathfrak{H}_0$ as $G_x$-Hilbert spaces for some Hilbert space $\mathfrak{H}_0$ with a trivial $G_x$-action ($G_x$ is the finite isotropy subgroup of $G$ at $x$). For operators acting on this Hilbert space, the notion of propagation is as before. The notion of local compactness, however, is replaced by that of *local compactness relative to* $A$ – meaning that the reductions are in $\mathcal{K}(\mathfrak{H}) \otimes A$.

A.4. **Definition** (Algebras with coefficients)**.** The Roe algebras with coefficients in $A$ are defined as follows:

(a) The Roe algebra with coefficients, $C^*(P_d(G), A)$, is the operator norm closure of $\mathbb{C}[X_d, A]$, the algebra of all finite propagation operators that are locally compact relative to $A$.
(b) The localization algebra with coefficients, $C^*_L(P_d(G), A)$ is defined as in (A.2).
(c) If $H$ is a subgroup of $G$, the $H$-equivariant Roe and localization algebras with coefficients, $C^*(P_d(G), A)^H$ and $C^*_L(P_d(G), A)^H$, are defined in analogy with (A.1) and (A.2), respectively.

Continuing, we have a local assembly map with coefficients analogous to (A.1); the local assembly map is related to the usual Baum-Connes assembly map via an evaluation homomorphism analogous to (A.2); the analog of Theorem A.3 remains valid.

The essential consequence of the above discussion is the following result.

A.5. **Theorem.** *The Baum-Connes and Novikov conjectures can be reformulated, replacing the term on the left by $K$-theory of the appropriate localization algebra. Precisely, the Baum-Connes conjecture with coefficients in $A$ is equivalent to the assertion that the evaluation map*

$$\mathrm{ev}_* : \lim_{d\to\infty} K_*(C^*_L(P_d(G), A)^G) \to \lim_{d\to\infty} K_*(C^*(P_d(G), A)^G).$$

*is an isomorphism; the Novikov conjecture is equivalent to the assertion that the evaluation map is injective.*

Of course, the utility of this result arises from the fact that it puts the left and right hand sides of the conjectures on the same footing. We illustrate with a final result, which is a version of the result of Chabert-Echterhoff [CE] sufficient for our earlier purpose.

A.6. **Theorem.** *Let $H$ be a finite index subgroup of $G$, with quotient $\Gamma$. We may identify the domains of assembly so that the diagram commutes:*

$$\begin{tikzcd}
KK^G_*(C_0(P_d(G)), \ell^\infty(\Gamma, \mathcal{K})) \arrow[r] \arrow[d, "\cong"] & K_*(C^*_r(G, \ell^\infty(\Gamma, \mathcal{K}))) \\
K^H_*(P_d(G)) \arrow[r] & K_*(C^*_r(H)) \arrow[u, "\cong"']
\end{tikzcd}$$

*Here, the isomorphism of $K$-theory groups on the right is implemented by the Morita equivalence* $C^*_r(H) \sim C^*_r(G, \ell^\infty(\Gamma, \mathcal{K}))$.

*Proof.* The essential points are the identifications

$$K_*(C_L^*(P_d(G),\ell^\infty(\Gamma,\mathcal{K}))^G) \cong K_*(C_L^*(P_d(G))^H),$$

and similarly for unlocalized algebras, which commute with the evaluation homomorphisms. These identifications arise from Morita equivalences of the underlying algebras. The unlocalized algebras are identified with crossed products using the following Morita equivalences,

$$C^*(|G|,\ell^\infty(\Gamma,\mathcal{K}))^G \sim C_r^*(G,\ell^\infty(\Gamma,\mathcal{K})), \quad C^*(|G|)^H \sim C_r^*(H),$$

the first of which generalizes Lemma 4.1 to incorporate coefficients. These identifications are compatible. Put together, we have the diagram:

$$\begin{tikzcd}
KK_*^G(C_0(P_d(G)),\ell^\infty(\Gamma,\mathcal{K})) \arrow[r, "\mu^G"] \arrow[dr, "\mu_L^G"'] & K_*(C^*(|G|,\ell^\infty(\Gamma,\mathcal{K}))^G) \arrow[r, "\cong"] & K_*(C_r^*(G,\ell^\infty(\Gamma,\mathcal{K}))) \arrow[ddd, leftrightarrow, "\cong"] \\
& K_*(C_L^*(P_d(G),\ell^\infty(\Gamma,\mathcal{K}))^G) \arrow[u, "ev_*^G"'] \arrow[d, leftrightarrow, "\cong"'] & \\
& K_*(C_L^*(P_d(G))^H) \arrow[d, "ev_*^H"'] & \\
K_*^H(P_d(G)) \arrow[r, "\mu^H"] \arrow[ur, "\mu_L^H"] & K_*(C^*(|G|)^H) \arrow[r, "\cong"] & K_*(C_r^*(H)).
\end{tikzcd}$$

The assembly maps are show to be factored through the appropriate local assembly maps, $\mu_L^G$ and $\mu_L^H$ which are isomorphisms; here we have made use of identifications

$$C^*(P_d(G),\ell^\infty(\Gamma,\mathcal{K}))^G \cong C^*(|G|,\ell^\infty(\Gamma,\mathcal{K}))^G, \quad C^*(P_d(G))^H \cong C^*(|G|)^H$$

to make sense of the evaluation homomorphisms. The composite across the bottom row is assembly for $H$ without coefficients [R2]; similarly, the top row is assembly for $G$ with coefficients. Using the diagram we identify the domains of assembly to finish. ☐

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