

WEAK AMENABILITY OF CAT(0)-CUBICAL GROUPS

ERIK GUENTNER AND NIGEL HIGSON

ABSTRACT. We prove that if G is a discrete group that admits a metrically proper action on a finite-dimensional CAT(0) cube complex X , then G is weakly amenable. We do this by constructing uniformly bounded Hilbert space representations π_z for which the quantities $z^{\ell(g)}$ are matrix coefficients. Here ℓ is a length function on G obtained from the combinatorial distance function on the complex X .

INTRODUCTION

Let G be a countable discrete group and denote by $C_\lambda^*(G)$ the reduced C^* -algebra of G . Every finitely supported function $\phi: G \rightarrow \mathbb{C}$ defines a bounded linear operator

$$M_\phi: C_\lambda^*(G) \rightarrow C_\lambda^*(G)$$

by the formula $(M_\phi f)(g) = \phi(g)f(g)$, as well as bounded linear operators

$$M_\phi^{(n)}: M_n(C_\lambda^*(G)) \rightarrow M_n(C_\lambda^*(G))$$

between matrix algebras by applying M_ϕ to each matrix entry. The *completely bounded multiplier norm* of ϕ is defined to be the quantity

$$\|\phi\|_{cb} = \sup_n \|M_\phi^{(n)}\|.$$

See [CH89, Pis95]. A countable discrete group is *weakly amenable* [CH89] if there exists a sequence of finitely supported functions on G converging pointwise to 1 and consisting of functions which are uniformly bounded in the completely bounded multiplier norm.¹ Recall that a countable discrete group G is *amenable* if and only if there exists a sequence of finitely supported, positive-definite functions on G converging pointwise to the constant function 1 on G . It may be shown that the completely bounded multiplier norm of a positive-definite function ϕ is equal to $\phi(e)$. As a result, every weakly amenable group is amenable.

The notion of amenability may be broadened in a different way. A c_0 -function on a discrete set is a function that extends to a continuous function on the one-point compactification with value 0 at infinity. A countable discrete group is said to have the *Haagerup property* [CCJ⁺01] if there exists a sequence of positive-definite c_0 -functions on G converging pointwise to the constant function 1 on G . Since every finitely supported function is a c_0 -function, every amenable group has the Haagerup property.

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¹This is not the standard definition, but see [Haa86, HK94].

Michael Cowling conjectures that every countable discrete group with the Haagerup property is weakly amenable [CCJ⁺01]. He further conjectures that the *Cowling-Haagerup constant* of any such group is 1, which means that there exists a sequence of finitely supported functions on G converging pointwise to 1 and consisting of functions whose completely bounded multiplier norm is bounded by 1. The conjecture is supported by the cases of Coxeter groups [Jan02] and discrete subgroups of $SO(n, 1)$ [dCH85] and $SU(n, 1)$ [Cow83].

In this paper we shall consider groups which admit an action on a finite-dimensional $CAT(0)$ -cubical complex which is proper in the sense that for every vertex x in the complex, the function $d(gx, x)$ on G is proper. We shall refer to these groups as $CAT(0)$ -cubical groups. Niblo and Reeves [NR97, NR98b] proved that $CAT(0)$ -cubical groups have the Haagerup property. We shall prove the following result:

Theorem. *Every $CAT(0)$ -cubical group is weakly amenable, with Cowling-Haagerup constant one.*

Our theorem adds evidence in favor of Cowling's conjecture. In fact it might be argued that the class of $CAT(0)$ -cubical groups is quite close to the class of all groups with the Haagerup property. It is known that a discrete group has the Haagerup property if and only if it acts properly on a *measured* space with walls (see [CMV04] for the precise assertion). Furthermore, a discrete group acts on a *combinatorial* space with walls if and only if it acts on a $CAT(0)$ cube complex, although possibly an infinite-dimensional $CAT(0)$ cube complex. See [CN05, Nic04]. Thus, disregarding the issue of infinite-dimensionality, the difference between the class of groups with the Haagerup property and the class of $CAT(0)$ -cubical groups comes down to the difference between combinatorial and measured spaces with walls. As Graham Niblo has explained to us, the distinction is analogous to that between trees and \mathbb{R} -trees.

Unfortunately the possible infinite-dimensionality of $CAT(0)$ cube complexes can pose serious problems. For example Farley [Far03] proved that Thompson's group F has the Haagerup property by exhibiting a metrically proper action of F on a certain $CAT(0)$ cube complex. However Farley's complex, although locally finite-dimensional, is infinite-dimensional and our theorem above does not apply. The question of whether or not F is weakly amenable remains open.

In order to prove the weak amenability of $CAT(0)$ -cubical groups we shall rely on a result of Valette [Val93] that makes it possible to deduce weak amenability from the existence of a suitable holomorphic family of uniformly bounded representations. Denote by \mathbb{D} the open unit disk in the complex plane. A family $\{\pi_z\}_{z \in \mathbb{D}}$ of representations of a discrete group G into the bounded invertible operators on a Hilbert space \mathcal{H} is *holomorphic* if for every element $g \in G$ and every pair of vectors $\xi, \xi' \in \mathcal{H}$ the matrix coefficient

$$\phi(z) = \langle \pi_z(g)\xi, \xi' \rangle$$

is a holomorphic function on \mathbb{D} . This definition is explored in [dCH85].

Theorem (See [Val93]). *Let G be a countable discrete group. Assume that there exists a proper function $\ell : \Gamma \rightarrow \mathbb{N}$ with $\ell(e) = 0$ and a holomorphic family $\{\pi_z\}_{z \in \mathbb{D}}$ of uniformly bounded representations of G on a Hilbert space \mathcal{H} such that π_t is unitary for $0 < t < 1$. If there exists a*

vector $\xi \in \mathcal{H}$ such that $z^{\ell(g)} = \langle \pi_z(g)\xi, \xi \rangle$ for every $z \in \mathbb{D}$ and every $g \in G$, then G is weakly amenable with Cowling-Haagerup constant one.

We shall construct the required representations following a method of Pimsner [Pim87] and Valette [Val90a] that we shall now outline. Let X be a set equipped with an action of G and let π be the natural permutation representation of G on $\ell^2(X)$. A function

$$c : X \times X \rightarrow \mathcal{B}(\ell^2(X))$$

is a *cocycle* for π if

- (a) $c(x, x) = 1$
- (b) $c(v, x)c(x, y) = c(v, y)$
- (c) $\pi(g)c(x, y)\pi(g)^{-1} = c(g \cdot x, g \cdot y)$

for all $v, x, y \in X$ and all $g \in G$. If c is a cocycle for π , and if $x \in X$, then the formula

$$\pi_c(g) = c(x, gx)\pi(g)$$

defines a representation of G into the bounded invertible operators on $\ell^2(X)$. If the cocycle c is *uniformly bounded*, meaning that

$$\sup_{x, y \in X} \|c(x, y)\| < \infty,$$

then π_c is uniformly bounded. If the cocycle is *unitary*, meaning that $c(x, y) = c(y, x)^*$ for all x and y , then the representation π_c is unitary.

We shall take X to be the set of vertices of a finite-dimensional CAT(0) cube complex on which G acts, and we shall construct a family of cocycles $\{c_z\}_{z \in \mathbb{D}}$ using the geometry of CAT(0) cube complexes. In doing so we shall closely follow Pimsner and Valette, who constructed uniformly bounded representations in this way when X is the set of vertices of a tree [Pim87, Val90a], and Januszkiewicz, who extended their results to products of trees [Jan93]. The same cocycle has also been analyzed very carefully by Brodzki, Niblo and Valette in unpublished work. The construction of the cocycle is straightforward; the proof that the cocycle is uniformly bounded is more difficult and this is our contribution.

1. OVERVIEW OF THE PROOF

To prepare the reader for the more complicated case of CAT(0) cube complexes, we shall rapidly review the construction of Pimsner and Valette for trees (which are simple examples of CAT(0) cube complexes). Let X be the set of vertices of a simplicial tree on which a group G acts. If $z \in \mathbb{D}$, then define $w \in \mathbb{C}$ by the equation $z^2 + w^2 = 1$, or $w = \sqrt{1 - z^2}$, where we use the branch of the square root function which is holomorphic on the complement of the negative real axis and which is positive on the positive real axis. If $x, y \in X$ are the vertices of an edge, then define $c_z(x, y) \in \mathcal{B}(\ell^2(X))$ by

$$c_z(x, y)\delta_v = \begin{cases} w\delta_x - z\delta_y, & v = x \\ w\delta_y + z\delta_x, & v = y \\ \delta_v, & \text{otherwise,} \end{cases}$$

where δ_v denotes the Dirac function at the vertex v . In other words, $c_z(x, y)$ is given by the matrix

$$\begin{pmatrix} w & z \\ -z & w \end{pmatrix}$$

on the two-dimensional subspace spanned by the ordered basis (δ_x, δ_y) and it is the identity on the orthogonal complement of this subspace. Notice that

$$(1.1) \quad c_z(x, y) = c_z(y, x)^{-1},$$

while if z is real, then $c_z(x, y) = c_z(y, x)^*$, and hence $c_z(x, y)$ is unitary. If x and y are arbitrary vertices in the tree, then define $c_z(x, y) \in \mathcal{B}(\ell^2(X))$ by

$$c_z(x, y) = c_z(v_0, v_1)c_z(v_1, v_2) \cdots c_z(v_{n-1}, v_n),$$

where $x = v_0, v_1, \dots, v_n = y$ are the vertices along any edge-path from x to y . It follows from (1.1) and the basic geometry of trees that this definition is independent of the path chosen between x and y , and it follows from this that c_z is a cocycle for π .

Fixing a vertex $x \in X$ we obtain representations $\pi_z(g) = c_z(x, gx)\pi_g$. This is a holomorphic family — indeed for each g , the operator $\pi_z(g)$ is a polynomial in z and w with coefficients in $\mathcal{B}(\ell^2(X))$. Moreover, if we define $\ell(g)$ to be $d(x, gx)$, the edge-path length between x and gx , then it is easy to check that $z^{\ell(g)} = \langle \pi_z(g)\delta_x, \delta_x \rangle$ (see for example the computations following Lemma 1.1 below). If z is real, then π_z is a unitary representation.

In order to apply Valette's theorem it remains to show that for general $z \in \mathbb{D}$, the representation π_z is uniformly bounded. We shall briefly indicate a series of lemmas sufficient to prove this that are representative of what we are able to accomplish for general CAT(0) cube complexes. Denote by c_{ab} the matrix entries of $c_z(x, y)$, so that if $b \in X$, then

$$c_z(x, y)\delta_b = \sum_{a \in X} c_{ab}\delta_a,$$

or equivalently $c_{ab} = \langle c_z(x, y)\delta_b, \delta_a \rangle$.

1.1. Lemma. *Each nonzero matrix entry c_{ab} has the form $c_{ab} = \pm z^k w^\ell$ where $k = d(a, b)$ and where $\ell \leq 2$.*

This is the geometric heart of the uniform boundedness argument. For trees it may be proved by explicitly computing all the coefficients c_{ab} . Let v_0, \dots, v_n be the vertices on the geodesic path from x to y . If b does not belong to this path, then it is easy to see that $c_z(x, y)\delta_b = \delta_b$. If $b = v_n$, then

$$\begin{aligned} c_z(x, y)\delta_b &= c_z(v_0, v_1) \cdots c_z(v_{n-1}, v_n)\delta_{v_n} \\ &= c_z(v_0, v_1) \cdots c_z(v_{n-2}, v_{n-1})(w\delta_{v_n} + z\delta_{v_{n-1}}) \\ &= c_z(v_0, v_1) \cdots c_z(v_{n-3}, v_{n-2})(w\delta_{v_n} + wz\delta_{v_{n-1}} + z^2\delta_{v_{n-2}}) \\ &\vdots \\ &= w\delta_{v_n} + wz\delta_{v_{n-1}} + wz^2\delta_{v_{n-2}} + \cdots + wz^{n-1}\delta_{v_1} + z^n\delta_{v_0} \end{aligned}$$

If $b = v_i$ with $i < n$, then $c_z(v_{i+1}, v_{i+2}) \cdots c_z(v_{n-1}, v_n) \delta_b = \delta_b$ and so

$$\begin{aligned} c_z(x, y) \delta_b &= c_z(v_0, v_1) \cdots c_z(v_i, v_{i+1}) \delta_{v_i} \\ &= c_z(v_0, v_1) \cdots c_z(v_{i-1}, v_i) (-z \delta_{v_{i+1}} + w \delta_{v_i}) \\ &= c_z(v_0, v_1) \cdots c_z(v_{i-2}, v_{i-1}) (-z \delta_{v_{i+1}} + w^2 \delta_{v_i} + zw \delta_{v_{i-1}}) \\ &\vdots \\ &= -z \delta_{v_{i+1}} + w^2 \delta_{v_i} + w^2 z \delta_{v_{i-1}} + \cdots + w^2 z^{n-i-1} \delta_{v_1} + wz^{n-i} \delta_{v_0}, \end{aligned}$$

The lemma follows by inspection of these formulas. In the general case of cube complexes such direct computations are not so easy, and we shall have to work harder to obtain weaker (but adequate) results. But proceeding with the argument for trees, we can now rapidly conclude that the cocycle c_z is uniformly bounded.

1.2. Lemma. *Let $x, y \in X$, and let $k \geq 0$. For every $b \in X$*

$$\# \{ a \in X : c_{ab} = \pm z^k w^\ell \text{ for some } \ell \text{ and all } z \in \mathbb{D} \} \leq 2$$

and for every $a \in X$

$$\# \{ b \in X : c_{ab} = \pm z^k w^\ell \text{ for some } \ell \text{ and all } z \in \mathbb{D} \} \leq 2.$$

Both assertions follow from the fact that for any given vertex v on a geodesic interval $[x, y]$ and any k , there are at most two vertices on the interval which are exactly distance k away from v .

1.3. Proposition. *For every $z \in \mathbb{D}$ and every $x, y \in X$ the quantity $\|c_z(x, y)\|$ is bounded by $4(1 - |z|)^{-1}$.*

Proof. Let $d(x, y) = N$. Decompose the operator $c_z(x, y)$ on the ℓ^2 -space of the interval $[x, y]$ as a sum

$$c_z^{(0)} + z c_z^{(1)} + z^2 c_z^{(2)} + \cdots + z^N c_z^{(N)},$$

where the operator $c_z^{(k)}$, contains only matrix entries of the form $\pm w^\ell$ or zero. Since $\ell \leq 2$, each matrix coefficient of $c_z^{(k)}$ absolute value 2, or less. Furthermore, by the previous lemma each row and column of $c_z^{(k)}$ has at most two non-zero entries. But a matrix which has at most A nonzero entries in each row and column, each of absolute value B or less, determines an operator of norm AB or less. Hence $\|c_z^{(k)}\| \leq 4$, for every $k \geq 0$, so that

$$\|c_z(x, y)\| \leq 4 \sum_{k=0}^N |z|^k,$$

and the proposition follows. \square

For cube complexes we shall obtain polynomial bounds in place of the absolute bounds in Lemma 1.2, but these will be adequate to carry through a version of the argument in the proof of Proposition 1.3.

In the case of trees, putting everything together, we obtain the following result (of Szwarc [Szw91] and Valette [Val90b]):

1.4. Theorem. *Suppose that a discrete group G acts on a tree and that for some vertex x in the tree the function $g \mapsto d(x, gx)$ is proper on X . Then G is weakly amenable with Cowling-Haagerup constant 1.* \square

2. CAT(0) CUBE COMPLEXES

In this section we shall rapidly review the notion of cube complex and collect the results about them that we shall need.

A *cube complex* [Gro87, BH99, NR98a] is a set X , called the set of *vertices*, together with a collection of finite subsets of X , called the *cubes* of X , such that: every single-element set is a cube; the intersection of any two cubes is a cube; and for every cube C , there is an integer $n \geq 0$ and a bijection from C to the vertices of the standard cube $[0, 1]^n$ in \mathbb{R}^n such that the cubes in X that are subsets of C correspond precisely to the sets of vertices of the faces (of all dimensions) of the standard cube. The cubes with 2 elements are called *edges*; those with 4 elements are *squares*. A cube complex X is *finite-dimensional* if its cubes are uniformly bounded in cardinality.

Every cube complex has a natural *geometric realization* in which each cube C is replaced by a unit cube in \mathbb{R}^n . This geometric realization has a natural metric which restricts to the standard metric on each Euclidean cube. A (globally) *non-positively curved cube complex*, or CAT(0) *cube complex* is a cube complex whose geometric realization is a CAT(0) metric space [BH99]. For a complex whose geometric realization is simply connected the non-positive curvature condition can be phrased combinatorially [Gro87, BH99].

The CAT(0) condition has important consequences which we shall access via the notion of hyperplane. A *hyperplane* in X is an equivalence class of edges under the equivalence relation generated by the relation

$$\{x, y\} \sim \{x', y'\} \iff \{x, x', y, y'\} \text{ is a square.}$$

This definition is a combinatorial proxy for the notion of *geometric hyperplane* in the geometric realization of X : a geometric hyperplane is the union of an equivalence class of midplanes of cubes under the equivalence relation generated by the relation

$$M_1 \sim M_2 \iff M_1 \cap M_2 \text{ is the midplane of a cube.}$$

See [NR98a] for more information and some helpful illustrations. The hyperplane associated to a geometrical hyperplane is the set of all edges bisected by the geometric hyperplane, and every hyperplane arises in this way. The geometric point of view suggests the following terminology: if an edge $\{x, y\}$ belongs to a hyperplane H , then we shall say that it *crosses* H ; that H *separates* x from y ; that x and y are *opposite* one another across H ; that x and y are *adjacent* to H ; and that H is adjacent to x and y .

The most significant consequence of the CAT(0) condition is that if X is a CAT(0) cube complex, then every geometric hyperplane separates the geometric resolution of X into precisely two path components [Sag95, Thm 4.10]. This can be expressed combinatorially as follows. A sequence of vertices v_0, \dots, v_n in X is an *edge-path* if each $\{v_i, v_{i+1}\}$ is an edge in X . The path *crosses* a hyperplane H if one of its edges belongs to H . Every hyperplane H in a CAT(0) cube

complex partitions the vertices of X into precisely two sets, such that every edge path from a vertex in one component to a vertex in the other crosses H , and such that every pair of vertices in either one of the components is connected by an edge path that does not cross H . These two sets are the *half-spaces* determined by H and will they be denoted H_{\pm} (this involves making an arbitrary choice of which half-space will be H_+ and which H_- ; having made such a choice we shall say that H is *oriented*). We shall say that the hyperplane H *separates* two vertices from one another if one vertex lies in each of the components H_{\pm} . If x and y are vertices of a CAT(0) cube complex X , then we shall denote by $\mathfrak{H}(x, y)$ the set of all hyperplanes that separate x from y .

The *distance* $d(x, y)$ between two vertices x and y of a cube complex X is the minimum length n of an edge-path v_0, \dots, v_n connecting them. An edge-path from x to y is *geodesic* if it achieves this minimum length.

2.1. Proposition ([Sag95], Thm. 4.13). *An edge-path in a CAT(0) cube complex from x to y crosses each hyperplane in $\mathfrak{H}(x, y)$. An edge-path from x to y is geodesic if and only if it crosses only the hyperplanes in $\mathfrak{H}(x, y)$, and crosses each one of those each exactly once.* \square

It follows that $d(x, y) = \#\mathfrak{H}(x, y)$.

2.2. Corollary. *Let x, y and v be vertices in a CAT(0) cube complex. The following are equivalent.*

- (a) v lies on a geodesic from x to y .
- (b) $d(x, y) = d(x, v) + d(v, y)$.
- (c) $\mathfrak{H}(x, v) \cap \mathfrak{H}(v, y) = \emptyset$.
- (d) $\mathfrak{H}(x, v) \cap \mathfrak{H}(v, y) = \emptyset$ and $\mathfrak{H}(x, y) = \mathfrak{H}(x, v) \cup \mathfrak{H}(v, y)$.
- (e) $\mathfrak{H}(x, v) \subseteq \mathfrak{H}(x, y)$.
- (f) $\mathfrak{H}(x, v) \subseteq \mathfrak{H}(x, y)$ and $\mathfrak{H}(v, y) \subseteq \mathfrak{H}(x, y)$.

\square

A *corner move* transforms an edge-path by changing a string u, v, w in the path into u, v', w , where $\{u, v, v', w\}$ is a square. A corner move does not alter the length of a edge-path or its endpoints.

2.3. Proposition ([Sag95], Thm. 4.6). *Any two geodesic edge-paths in a CAT(0) cube complex with the same endpoints differ by a sequence of corner moves.* \square

A *simple cancellation* in an edge-path replaces a string v, v', v by the singleton string v .

2.4. Proposition. *Any two paths in a CAT(0) cube complex with the same endpoints are related by a sequence of corner moves and simple cancellations.*

To prove this we shall need an additional fact about hyperplanes in CAT(0) cube complexes. Two hyperplanes H and K *intersect* if their geometric realizations intersect, or equivalently if there is a square $\{x, y, z, w\}$ such that $\{x, y\}$ and $\{z, w\}$ belong to H while $\{x, z\}$ and $\{y, w\}$ belong to K .

2.5. Lemma ([NR98a], Prop. 2.10). *A hyperplane in a CAT(0) cube complex does not self-intersect. In fact if a vertex x is adjacent to a hyperplane H , then there is a unique edge $\{x, y\}$ which crosses H .* \square

We shall use the notation $y = x^{\text{op}}$ for the vertex y opposite x across H (obviously this depends on the choice of H).

Proof of Proposition 2.4. By the Proposition 2.3 it suffices to show that any edge-path which is not a geodesic can be reduced to an edge-path of shorter length by a sequence of corner moves and simple cancellations. Assume, for the sake of a contradiction, that not every non-geodesic edge-path is so reducible and let v_0, \dots, v_n be an non-geodesic edge-path of minimal length that is not reducible. Since it is not a geodesic it must cross some hyperplane H twice. By minimality, it must do so between v_0 and v_1 , and then again between v_{n-1} and v_n . Moreover the edge-path v_1, \dots, v_{n-1} must be a geodesic. Therefore by [Sag95, Thm. 4.13], all the vertices v_2, \dots, v_{n-1} are adjacent to H . If $n = 2$, then both v_0 and v_n are opposite v_1 across H , and so $v_0 = v_n$ by Lemma 2.5. The path v_0, v_1, v_2 is therefore v_0, v_1, v_0 and hence is reducible by a simple cancellation, which is a contradiction. If $n > 2$, then by [Sag95, Thm. 4.12] the subset $\{v_0, v_1, v_2, v_2^{\text{op}}\}$ is a square. The path $v_0, v_2^{\text{op}}, v_2, \dots, v_n$ is obtained from v_0, \dots, v_n by a corner move. But the path $v_2^{\text{op}}, v_2, \dots, v_n$ has length $n - 1$ and is not a geodesic (it crosses H twice). It is therefore reducible by corner moves and simple cancellations. It follows that the path $v_0, v_2^{\text{op}}, v_2, \dots, v_n$ is reducible as well, and hence so is v_0, \dots, v_n , which is again a contradiction. \square

The *convex hull* $\mathfrak{C}(S)$ of a set S of vertices in X is the intersection of all the half-spaces that contain S . A set is *convex* if it is equal to its convex hull. It follows from Proposition 2.1 that every convex set is geodesically convex in the sense that it contains every geodesic between any two of its points. In the case where $S = \{x, y\}$, the convex hull is usually called the *interval* from x to y and thanks to Corollary 2.2 it can alternately be characterized as the set of all points that lie on geodesics from x to y .

In Section 5 we shall need the following technical result:

2.6. Lemma. *Let x and y be vertices in a $\text{CAT}(0)$ cube complex and let b be a vertex of some cube C that also contains y as a vertex. There is a vertex c of C such that $\mathfrak{C}(x, y, b) \subseteq \mathfrak{C}(x, c)$.*

Proof. Let \mathfrak{H} be the set of all hyperplanes, necessarily passing through the cube C , that separate b from y but not x from y . Let c be the vertex of C for which $\mathfrak{H}(y, c) = \mathfrak{H}$. We shall show that

$$\mathfrak{C}(x, y, b) \subseteq \mathfrak{C}(x, c).$$

To do so, we must show that $y \in \mathfrak{C}(x, c)$ and $b \in \mathfrak{C}(x, c)$. By construction of the vertex c , $\mathfrak{H}(x, y)$ is disjoint from $\mathfrak{H}(y, c)$. It therefore follows from Corollary 2.2 that y lies on a geodesic from x to c and hence that $y \in \mathfrak{C}(x, c)$. Similarly, the definition of $\mathfrak{H} = \mathfrak{H}(x, c)$ asserts that $\mathfrak{H}(y, c) = \mathfrak{H}(b, c) \setminus \mathfrak{H}(x, y)$, and therefore

$$\mathfrak{H}(b, c) \subseteq \mathfrak{H}(x, y) \cup \mathfrak{H}(y, c).$$

Since y lies on a geodesic from x to c , the union is $\mathfrak{H}(x, c)$. Therefore $\mathfrak{H}(b, c) \subseteq \mathfrak{H}(x, c)$, and so b lies on a geodesic from x to c , and hence is in the interval from x to c . \square

The next three results have to do with multiple hyperplanes that are adjacent to a single vertex.

2.7. Proposition ([NR98a], Lemma 2.14). *If two hyperplanes in a CAT(0) cube complex intersect and are both adjacent to the same vertex, then they intersect in a square containing that vertex.* \square

2.8. Proposition. *Let H_1, \dots, H_n be hyperplanes in a CAT(0) cube complex, all adjacent to a vertex x . If there is a vertex y in X such that each hyperplane H_i separates x from y , then there is a cube of dimension n in which all the hyperplanes H_i intersect.*

Proof. This follows from Proposition 3.3 in [NR98a] (compare the remark following the proof of that proposition). \square

2.9. Proposition. *Let x and y be vertices of CAT(0) cube complex. There exists a geodesic path from x to y that crosses all the hyperplanes in $\mathfrak{H}(x, y)$ that are adjacent to x before it crosses any other hyperplane.*

Proof. By Proposition 2.8, there is a cube C of dimension n containing x as a vertex in which the n hyperplanes in $\mathfrak{H}(x, y)$ that are adjacent to x intersect. Let v be the vertex diagonally across C from x . By the equivalence of (a) and (e) in Corollary 2.2, there is a geodesic from x to y that passes through v , and any such geodesic has the required property. \square

Finally, two hyperplanes are *parallel* if they do not intersect.

2.10. Lemma. *Two parallel hyperplanes separate a CAT(0) cube complex into at most three² components.*

Proof. It is convenient to work in the geometric realization. Let H and K be parallel hyperplanes. Since K is connected, it is contained entirely in one component H_+ of the complement of H . It follows that one of K_+ or K_- does not intersect H_- , for if both were to intersect H_- , then there would be a geodesic between a point in K_+ and a point in K_- within H_- , and hence a point of K within H_- . If say $K_- \cap H_- = \emptyset$, then

$$\begin{aligned} X \setminus (H \cup K) &= (H_+ \cup H_-) \cap (K_+ \cup K_-) \\ &= (H_+ \cap K_+) \cup (H_+ \cap K_-) \cup (H_- \cap K_+), \end{aligned}$$

as required. \square

3. COCYCLES FROM CUBE COMPLEXES

An *action* of a group on a cube complex X is an action on the set of vertices of X that maps cubes to cubes. In this section we shall associate to the action of a group G on a CAT(0) cube complex X a holomorphic family of cocycles $c_z: X \times X \rightarrow \mathcal{B}(\ell^2(X))$ parametrized by $z \in \mathbb{D}$.

As in Section 1, we associate to each $z \in \mathbb{D}$ a complex number $w \in \mathbb{C}$ satisfying $z^2 + w^2 = 1$ in such a way that w depends holomorphically on z . Let x and y be adjacent vertices in X and let H be the hyperplane that separates them, oriented so that $x \in H_+$ and $y \in H_-$. Let us denote by

²In fact exactly three.

∂H_+ the set of vertices adjacent to H that lie in H_+ , and by ∂H_- the set of vertices adjacent to H that lie in H_- . Define a bounded operator $c_z(x, y)$ on $\ell^2(X)$ by means of the formula

$$(3.1) \quad c_z(x, y)\delta_v = \begin{cases} w\delta_v - z\delta_{v^{op}} & v \in \partial H_+ \\ w\delta_v + z\delta_{v^{op}} & v \in \partial H_- \\ \delta_v & v \notin \partial H_+ \cup \partial H_- \end{cases}$$

(where v^{op} denotes the vertex adjacent to v across H). The definition is consistent with the one we made for trees in Section 1. However in the present, more general situation $c_z(x, y)$ is potentially nontrivial on many more basis vectors than δ_x and δ_y alone: it is nontrivial on the basis vector δ_v whenever v is adjacent to the hyperplane H .

Recall that a subspace of a Hilbert space is a *reducing subspace* for an operator T if and only if both the subspace and its orthogonal complement are invariant under T . If the vertices v and v^{op} are adjacent across H , then the two-dimensional subspace spanned by δ_v and $\delta_{v^{op}}$ is reducing for $c_z(x, y)$, as is the subspace spanned by all δ_v for which v is *not* adjacent to v . Thus by Lemma 2.5, $c_z(x, y)$ is the direct sum of a family of operators on two-dimensional subspaces together with the identity operator on the joint orthogonal complement of these two-dimensional subspaces. On the two-dimensional subspace spanned by the ordered pair $(\delta_v, \delta_{v^{op}})$, where $v \in \partial H_+$, the operator $c_z(x, y)$ is given by the matrix

$$(3.2) \quad \begin{pmatrix} w & z \\ -z & w \end{pmatrix}.$$

3.1. Lemma. *For every pair of adjacent edges x and y , and for every $z \in \mathbb{D}$, $c_z(x, y)c_z(y, z) = I$.*

Proof. This follows from the fact that the inverse of the matrix (3.2) is the transpose of (3.2). \square

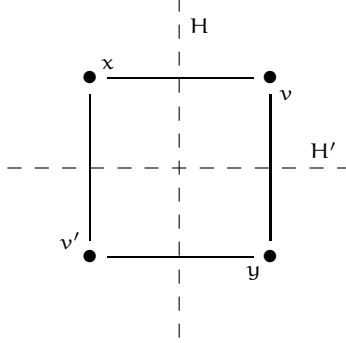
If x and y are any two vertices in X , then we should like to define

$$(3.3) \quad c_z(x, y) = c_z(v_0, v_1)c_z(v_1, v_2) \cdots c_z(v_{n-1}, v_n),$$

where $x = v_0, \dots, v_n = y$ are the vertices in an edge-path from x to y . In the case of trees, Lemma 3.1 is sufficient to prove that the product (3.3) is independent of the choice of path from x to y . For general CAT(0) cube complexes we need the following additional computation.

3.2. Lemma. *Let x, v, y and x, v', y be edge-paths of length two from x to y . If the four vertices x, v, v', y span a square in the cube complex X , then $c_z(x, v)c_z(v, y) = c_z(x, v')c_z(v', y)$.*

Proof. The relation between the four points is depicted in the following diagram:



The hyperplane H separating x and v also separates v' and y , while the hyperplane H' separating x and v' also separates v and y . As result $c_z(x, v) = c_z(v', y)$ and $c_z(x, v') = c_z(v, y)$, since the definition of c_z for a pair of adjacent vertices depends only on the oriented hyperplane separating them. So our goal is to show that

$$c_z(x, v)c_z(x, v') = c_z(x, v')c_z(x, v).$$

By Lemmas 2.5 and 2.7 the set X can be written as a disjoint union of:

- (a) Four-element sets consisting of the vertices of a square in which H and H' intersect.
- (b) Two-element sets consisting of vertices adjacent across H but with neither adjacent to H' .
- (c) Two-element sets consisting of vertices adjacent across H' but with neither adjacent to H .
- (d) Single-element sets consisting of a vertex adjacent to neither H nor H' .

The corresponding subspaces of $\ell^2(X)$ are reducing for $c_z(x, v)$ and $c_z(x, v')$. On the four-dimensional subspaces, the prototype of which is the one spanned by $\delta_x, \delta_v, \delta_{v'}, \delta_y$, the operators $c_z(x, v)$ and $c_z(x, v')$ are represented by matrices

$$\begin{pmatrix} w & z & 0 & 0 \\ -z & w & 0 & 0 \\ 0 & 0 & w & z \\ 0 & 0 & -z & w \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w & 0 & z & 0 \\ 0 & w & 0 & z \\ -z & 0 & w & 0 \\ 0 & -z & 0 & w \end{pmatrix},$$

and these two matrices commute. On the remaining subspaces, either one or both of $c_z(x, v)$ and $c_z(x, v')$ acts as the identity, and so the two operators commute there too. \square

3.3. Lemma. *The expression (3.3) defining $c(x, y)$ for general $x, y \in X$ is independent of the edge-path v_0, v_1, \dots, v_n connecting x to y .*

Proof. According to Proposition 2.4 any two paths connecting x to y are related by a sequence of corner moves and simple cancellations. Lemma 3.1 shows that simple cancellations do not alter (3.3). Lemma 3.2 shows that corner moves do not alter (3.3) either. \square

3.4. Proposition. *Let X be a $\text{CAT}(0)$ cube complex and let G be a discrete group acting on X . For every $z \in \mathbb{D}$ the function $c_z : X \times X \rightarrow \mathcal{B}(\ell^2(X))$ defined by (3.3) is a cocycle for the permutation representation of G on $\ell^2(X)$.* \square

Fix $x \in X$. Thanks to the proposition we can construct representations $\pi_z(g) = c_z(x, gx)\pi(g)$ of the group G into the bounded invertible operators on $\ell^2(X)$.

3.5. Proposition. *The family $\{\pi_z\}_{z \in \mathbb{D}}$ is a holomorphic family of representations of G . If $z \in \mathbb{D}$ is real, then the representation π_z is unitary.* \square

3.6. Proposition. *The matrix coefficient $\langle \pi_z(g)\delta_x, \delta_x \rangle$ is equal to $z^{d(x, gx)}$.*

Proof. By definition, $\langle \pi_z(g)\delta_x, \delta_x \rangle = \langle c_z(x, gx)\delta_{gx}, \delta_x \rangle$, so it suffices to prove that

$$(3.4) \quad \langle c_z(x, y)\delta_y, \delta_x \rangle = z^{d(x, y)}$$

for every x and y in X . Note first that by definition $c_z(x, y)$ can be written as a product of $d(x, y)$ many operators $T = c_z(v_i, v_{i+1})$, each of which has the property that $\langle T\delta_a, \delta_b \rangle = 0$ if $d(a, b) > 1$. It follows that

$$(3.5) \quad \langle c_z(x, y)\delta_a, \delta_b \rangle = 0 \quad \text{if } d(a, b) > d(x, y).$$

We can now prove the formula (3.4) by induction on $d(x, y)$. Let $x = v_0, v_1, \dots, v_n = y$ be a geodesic edge-path from x to y . Using (3.5) and the induction hypothesis we have that

$$c_z(v_1, v_n)\delta_{v_n} = 0 \cdot \delta_{v_0} + z^{n-1}\delta_{v_1} + \text{terms orthogonal to } \delta_{v_0}, \delta_{v_1}.$$

Using the explicit formula for $c_z(v_0, v_1)$ we get that

$$c_z(v_0, v_n)\delta_{v_n} = c_z(v_0, v_1)c_z(v_1, v_n)\delta_{v_n} = z^n\delta_{v_0} + \text{terms orthogonal to } \delta_{v_0},$$

as required. \square

4. CALCULATION OF MATRIX COEFFICIENTS

The remainder of the paper will be devoted to proving that the cocycle c_z is uniformly bounded. Following the approach we took for trees in Section 1, we shall begin by studying the individual matrix coefficients of the operator $c_z(x, y)$. Define the matrix coefficient c_{ab} by the formula

$$c_z(x, y)\delta_b = \sum_{a \in X} c_{ab}\delta_a,$$

or equivalently $c_{ab} = \langle c_z(x, y)\delta_b, \delta_a \rangle$.

4.1. Lemma. *Let x and y be any two vertices in X . If K is a hyperplane which does not separate x from y , then the subspaces $\ell^2(K_{\pm})$ are reducing subspaces for $c_z(x, y)$. Hence if c_{ab} is non-zero for some $z \in \mathbb{D}$, then $\mathfrak{H}(a, b) \subseteq \mathfrak{H}(x, y)$.*

Proof. Assume first that x and y are adjacent. The only nonzero matrix coefficients of $c_z(x, y)$ apart from those on the diagonal are those c_{ab} for which a and b are adjacent to one another across the hyperplane H separating x from y . So if c_{ab} is nonzero for some $z \in \mathbb{D}$, then a and b must lie on opposite sides of H , and hence on the same side of K . In general, let v_0, \dots, v_n be a geodesic edge-path from x to y , so that

$$c_z(x, y) = c_z(v_0, v_1)c_z(v_1, v_2) \cdots c_z(v_{n-1}, v_n).$$

The hyperplane K separates no v_i from v_{i+1} . Therefore the subspaces $\ell^2(K_\pm)$ are reducing for each $c_z(v_{i-1}, v_i)$, and hence for $c_z(x, y)$ as well. \square

4.2. Proposition. *Let x, y and b be vertices of X . If c_{ab} is nonzero for some $z \in \mathbb{D}$, then a lies in the convex hull of $\{x, y, b\}$.*

Proof. Assume that c_{ab} is nonzero for some $z \in \mathbb{D}$. Let H be a hyperplane for which all of x, y and b lie in the same half-space. If a lies in the other half-space then $H \in \mathfrak{H}(a, b)$ but not in $\mathfrak{H}(x, y)$ contradicting Lemma 4.1. \square

4.3. Definition. Let $x, y \in X$. A *geodesic order* on $\mathfrak{H}(x, y)$ is a linear order for which there exists a geodesic edge-path from x to y such that $H' < H''$ if and only if the path crosses H' before it crosses H'' .

4.4. Lemma. *If c_{ab} is nonzero for some $z \in \mathbb{D}$, then every geodesic order on $\mathfrak{H}(x, y)$ induces a geodesic order on $\mathfrak{H}(a, b)$.*

Proof. We shall prove the lemma by induction on the integer $d(x, y)$ (starting with $d(x, y) = 0$, where the result is trivial). Assume that $c_{ab} \neq 0$, for some $z \in \mathbb{D}$. Let $\{H_1, \dots, H_n\}$ be a geodesic order on $\mathfrak{H}(x, y)$ and let v_0, \dots, v_n be the corresponding geodesic path from x to y , so that

$$c_z(x, y) = c_z(v_0, v_1)c_z(v_1, v_2) \cdots c_z(v_{n-1}, v_n) = c_z(v_0, v_1)c_z(v_1, v_n).$$

To prove that the given geodesic order on $\mathfrak{H}(x, y)$ restricts to a geodesic order on $\mathfrak{H}(a, b)$ we shall consider two cases. The first is that a is not adjacent to the hyperplane H_1 that separates v_0 from v_1 . In this case, since

$$(4.1) \quad c_{ab} = \langle c_z(v_1, v_n)\delta_b, c_z(v_0, v_1)^*\delta_a \rangle,$$

and since $c_z(v_0, v_1)^*\delta_a = \delta_a$, the ab -matrix coefficient for $c_z(x, y)$ is equal to the ab -matrix coefficient for $c_z(v_1, v_n)$ and in particular the latter is nonzero. By the induction hypothesis, the given geodesic order on $\mathfrak{H}(v_1, v_n)$ restricts to a geodesic order on $\mathfrak{H}(a, b)$. But this order on $\mathfrak{H}(a, b)$ is the same as the order restricted from $\mathfrak{H}(x, y)$.

In the second case, a is adjacent to H_1 . Denote by a^{op} the vertex adjacent to a across H_1 . From (4.1) and the definition of $c_z(v_0, v_1)$ we get that

$$(4.2) \quad c_{ab} = w \cdot \langle c_z(v_1, v_n)\delta_b, \delta_a \rangle \pm z \cdot \langle c_z(v_1, v_n)\delta_b, \delta_{a^{\text{op}}} \rangle.$$

If H_1 separates a from b then by Lemma 4.1 the first inner product in (4.2) is zero, and hence

$$c_{ab} = \pm z \cdot \langle c_z(v_1, v_n)\delta_b, \delta_{a^{\text{op}}} \rangle.$$

Therefore the $a^{\text{op}}b$ -matrix coefficient for $c_z(v_1, v_n)$ is nonzero. By the induction hypothesis, the given geodesic order on $\mathfrak{H}(v_1, v_n)$ restricts to a geodesic order on $\mathfrak{H}(a^{\text{op}}, b)$. Since

$$\mathfrak{H}(a, b) = \{H_1\} \cup \mathfrak{H}(a^{\text{op}}, b) \quad \text{and} \quad \mathfrak{H}(x, y) = \{H_1\} \cup \mathfrak{H}(v_1, v_n),$$

it follows easily that the geodesic order on $\mathfrak{H}(x, y)$ restricts to a geodesic order on $\mathfrak{H}(a, b)$. If H_1 does *not* separate a from b , then it separates a^{op} from b . The second inner product in (4.2) is therefore zero, and hence

$$c_{ab} = w \cdot \langle c_z(v_1, v_n) \delta_b, \delta_a \rangle,$$

so that ab -matrix coefficient for $c_z(v_1, v_n)$ is nonzero. By the induction hypothesis, the geodesic order on $\mathfrak{H}(v_1, v_n)$ restricts to a geodesic order on $\mathfrak{H}(a, b)$. This immediately implies that the given geodesic order on $\mathfrak{H}(x, y)$ restricts to a geodesic order on $\mathfrak{H}(a, b)$. \square

4.5. Lemma. *A linear ordering $\{H_1, \dots, H_n\}$ on $\mathfrak{H}(x, y)$ is a geodesic ordering if and only if the vertex $v_0 = x$ is adjacent to H_1 and for each $i = 1, \dots, n$ the vertex v_i obtained by successively reflecting v_0 across H_1, \dots, H_{i-1} is adjacent to H_i . In this case the sequence of vertices v_0, \dots, v_n is a geodesic edge-path from x to y .*

Proof. If a linear ordering on $\mathfrak{H}(x, y)$ is induced from a geodesic edge-path from x to y , then the sequence v_0, \dots, v_n is precisely the sequence of vertices along the path, and so the adjacency condition is satisfied. Conversely, if the adjacency condition is satisfied, then v_0, \dots, v_n is a geodesic edge-path from x to v_n . The vertex v_n must equal y since any hyperplane K separating the two points would separate either v_n from $v_0 = x$ or separate x from y . In fact by construction of the path v_0, \dots, v_n , the hyperplane K would necessarily separate *both* v_n and y from $v_0 = x$, which is a contradiction since K could not then separate v_n from y . \square

4.6. Lemma. *Let $x, y, a, b \in X$ and suppose that no hyperplane in $\mathfrak{H}(x, y)$ separates b from a . Then $\langle c_z(x, y) \delta_b, \delta_a \rangle = w^\ell \langle \delta_b, \delta_a \rangle$, where ℓ is the number of hyperplanes in $\mathfrak{H}(x, y)$ that are adjacent to a .*

Proof. The proof is by induction on $n = d(x, y)$. The case $n = 0$ is trivial, so assume that $n > 0$. Let v_0, \dots, v_n be a geodesic edge-path from x to y and let H be the hyperplane that separates v_{n-1} from v_n . Then

$$c_z(v_{n-1}, v_n) \delta_b = \begin{cases} w \delta_b \pm z \delta_{b^{\text{op}}} & \text{if } b \text{ is adjacent to } H \\ \delta_b & \text{otherwise,} \end{cases}$$

where b^{op} is the vertex adjacent to b across H . Since $\ell^2(H_\pm)$ are reducing subspaces for $c_z(v_0, v_{n-1})$, it follows that $c_z(v_0, v_{n-1}) \delta_{b^{\text{op}}}$ is orthogonal to δ_a . As a result

$$\begin{aligned} \langle c_z(v_0, v_n) \delta_b, \delta_a \rangle &= \langle c_z(v_0, v_{n-1}) c_z(v_{n-1}, v_n) \delta_b, \delta_a \rangle \\ &= \begin{cases} w \langle c_z(v_0, v_{n-1}) \delta_b, \delta_a \rangle & \text{if } b \text{ is adjacent to } H \\ \langle c_z(v_0, v_{n-1}) \delta_b, \delta_a \rangle & \text{otherwise.} \end{cases} \end{aligned}$$

The result follows. \square

Suppose that c_{ab} is non-zero for some $z \in \mathbb{D}$. Fix a geodesic order $\{H_1, \dots, H_n\}$ on $\mathfrak{H}(x, y)$, and let v_0, \dots, v_n be the corresponding geodesic edge-path from x to y . Let

$$\mathfrak{H}(a, b) = \{H_{n_1}, \dots, H_{n_p}\},$$

and let a_0, \dots, a_p be the geodesic edge-path from a to b , guaranteed by Lemma 4.4, that corresponds to this ordering of $\mathfrak{H}(a, b)$. Note that $d(a, b) = p$.

For $j = 0, \dots, p$, let ℓ_j be the number of hyperplanes in $\{H_k : n_j < k < n_{j+1}\}$ that are adjacent to a_j (for convenience we are setting $n_0 = 0$ and $n_{p+1} = n + 1$).

4.7. Lemma. *With the above notation, $c_{ab} = \pm z^p w^{\ell_0 + \dots + \ell_p}$.*

Proof. We shall prove by induction on j , from $j = p$ down to $j = 0$, that

$$(4.3) \quad \langle c_z(v_{n_j}, v_n) \delta_b, \delta_{a_j} \rangle = \pm z^{p-j} w^{\ell_j + \dots + \ell_p}.$$

The case $j = p$ is a consequence of Lemma 4.6, while the assertion in the current lemma is the case $j = 0$. Assume that (4.3) holds for a given j . To compute (4.3) with $j - 1$ in place of j we shall write

$$c_z(v_{n_{j-1}}, v_n) \delta_b = c_z(v_{n_{j-1}}, v_{n_j-1}) c_z(v_{n_j-1}, v_n) \delta_b$$

and then write $c_z(v_{n_j-1}, v_{n_{p+1}}) \delta_b$ as a finite linear combination

$$c_z(v_{n_j-1}, v_n) \delta_b = \sum \alpha_k \delta_{b_k}.$$

As long as the sum contains no zero terms, it follows from Lemma 4.1 that every vertex b_k lies in the same half-space of every $H \in \mathfrak{H}(v_{n_{j-1}}, v_{n_j-1})$ as the vertex b . Moreover b and a_{j-1} lie in the same half-spaces of these hyperplanes because the only hyperplanes in $\mathfrak{H}(x, y)$ that separate a_{j-1} from $b = a_p$ are H_{n_j}, \dots, H_{n_p} . It therefore follows from Lemma 4.6 that

$$\begin{aligned} \langle c_z(v_{n_{j-1}}, v_n) \delta_b, \delta_{a_{j-1}} \rangle &= \sum \alpha_k \langle c_z(v_{n_{j-1}}, v_{n_j-1}) \delta_{b_k}, \delta_{a_{j-1}} \rangle \\ &= w^{\ell_{j-1}} \sum \alpha_k \langle \delta_{b_k}, \delta_{a_{j-1}} \rangle \\ &= w^{\ell_{j-1}} \langle c_z(v_{n_j-1}, v_n) \delta_b, \delta_{a_{j-1}} \rangle. \end{aligned}$$

In addition,

$$\langle c_z(v_{n_j-1}, v_n) \delta_b, \delta_{a_{j-1}} \rangle = \langle c_z(v_{n_j}, v_n) \delta_b, c_z(v_{n_j-1}, v_{n_j})^* \delta_{a_{j-1}} \rangle$$

and

$$\langle c_z(v_{n_j}, v_n) \delta_b, c_z(v_{n_j-1}, v_{n_j})^* \delta_{a_{j-1}} \rangle = \pm \langle c_z(v_{n_j}, v_n) \delta_b, \bar{z} \delta_{a_j} \rangle,$$

by definition of the geodesic path a_0, \dots, a_p . We conclude that

$$\langle c_z(v_{n_{j-1}}, v_n) \delta_b, \delta_{a_{j-1}} \rangle = \pm z^{p-(j-1)} w^{\ell_{j-1} + \dots + \ell_p}$$

as required. \square

4.8. Lemma. *If $x, y, a, b \in X$, then $\mathfrak{H}(x, y) \setminus \mathfrak{H}(a, b) \subset \mathfrak{H}(a, x) \triangle \mathfrak{H}(b, y)$ (symmetric difference).*

Proof. Let $H \in \mathfrak{H}(x, y) \setminus \mathfrak{H}(a, b)$. Then x and y are on opposite sides of H while a and b are on the same side. Either a and b lie on the side containing y , in which case $H \in \mathfrak{H}(a, x) \setminus \mathfrak{H}(b, y)$ or they lie on the side containing x , in which case $H \in \mathfrak{H}(b, y) \setminus \mathfrak{H}(a, x)$. \square

4.9. Proposition. *Let x and y be vertices of X . As usual, for $a, b \in X$ let $c_{ab} = \langle c_z(x, y)\delta_b, \delta_a \rangle$. If c_{ab} is nonzero for some $z \in \mathbb{D}$, then*

$$c_{ab} = \pm z^{d(a,b)} w^\ell,$$

for some non-negative integer ℓ not exceeding twice the dimension of X .

Proof. In view of Lemma 4.7 we need only show that the sum $\ell_0 + \dots + \ell_p$ appearing there is bounded by twice the dimension of X . Recall that ℓ_j is the cardinality of the set \mathfrak{H}_j of hyperplanes in $\{H_k : n_j < k < n_{j+1}\}$ that are adjacent to a_j . The only hyperplanes in $\mathfrak{H}(x, y)$ that separate a from b are H_{n_1}, \dots, H_{n_p} . Therefore

$$\mathfrak{H}_0 \cup \dots \cup \mathfrak{H}_p \subseteq \mathfrak{H}(x, y) \setminus \mathfrak{H}(a, b),$$

and so by Lemma 4.8,

$$\mathfrak{H}_0 \cup \dots \cup \mathfrak{H}_p \subseteq \mathfrak{H}(a, x) \triangle \mathfrak{H}(b, y).$$

We shall show that if H belongs to both $\mathfrak{H}_0 \cup \dots \cup \mathfrak{H}_p$ and $\mathfrak{H}(a, x) \setminus \mathfrak{H}(b, y)$, then H is adjacent to the vertex a . Similarly, we shall show that if H belongs to both $\mathfrak{H}_0 \cup \dots \cup \mathfrak{H}_p$ and $\mathfrak{H}(b, y) \setminus \mathfrak{H}(a, x)$, then H is adjacent to the vertex b . It will follow from this that $\mathfrak{H}_0 \cup \dots \cup \mathfrak{H}_p$ is a union of two sets, the first consisting of hyperplanes adjacent to a that separate a from x , and the second consisting of hyperplanes adjacent to b that separate b from y . Since Proposition 2.8 implies that the hyperplanes in each set meet in a cube of X , it will follow that each set can have at most $\dim(X)$ elements, and therefore that $\ell_0 + \dots + \ell_p$ is bounded by $2 \dim(X)$, as required.

Let $H \in \mathfrak{H}_j$. Thus $H = H_k$, where $n_j < k < n_{j+1}$, and H is adjacent to a_j . Assume in addition that $H \in \mathfrak{H}(a, x) \setminus \mathfrak{H}(b, y)$. We shall show that a and indeed the entire geodesic edge-path a_0, \dots, a_j is adjacent to H by proving that if a vertex a_s on this path is adjacent to H , and if $s > 1$, then a_{s-1} is adjacent to H too.

Let K be the hyperplane that separates a_{s-1} from a_s (thus $K = H_{n_s}$.) Orient the hyperplanes H and K so that $b \in H_+ \cap K_+$. Then $a \in H_+ \cap K_-$ since $K \in \mathfrak{H}(a, b)$ while $H \notin \mathfrak{H}(a, b)$. In addition $y \in H_+$ and $x \in H_-$ since $H \notin \mathfrak{H}(b, y)$. Since $K < H$ in the geodesic order from x to y , there is at least one vertex v on the geodesic v_0, \dots, v_n that K but not H separates from x . If $x \in K_-$, then $x \in H_- \cap K_-$ and $v \in H_- \cap K_+$, while if $x \in K_+$, then $x \in H_- \cap K_+$ and $v \in H_- \cap K_-$. In either case, all four of the half-space intersections $H_\pm \cap K_\pm$ are nonempty, and it therefore follows from Lemma 2.10 that H and K intersect. Since a_s is adjacent to both H and K , it follows from Proposition 2.7 that H and K intersect in a square having a_s as a vertex, and this implies that a_{s-1} is adjacent to H , as required.

The proof that if H belongs to both $\mathfrak{H}_0 \cup \dots \cup \mathfrak{H}_p$ and $\mathfrak{H}(b, y) \setminus \mathfrak{H}(a, x)$, then H is adjacent to the vertex b is exactly the same. \square

5. UNIFORM BOUNDEDNESS OF THE COCYCLE

Our proof that the operators $c_z(x, y)$ are uniformly bounded as x and y range over all of X (while z ranges over a compact subset of \mathbb{D}) will be based on the results of the previous section and the following estimate of Chatterji and Ruane [CR05] (we shall offer our own proof of the estimate in the appendix).

5.1. Proposition. *Let X be a finite-dimensional CAT(0) cube complex and let x, y be vertices of X . If $k \geq 0$, then denote by $B(y, k)$ the set of vertices of X of distance k or less to y . Then*

$$\#(\mathfrak{C}(x, y) \cap B(y, k)) \leq (k + 1)^d,$$

where $d = \dim(X)$. □

5.2. Lemma. *Let $x, y \in X$. There is a polynomial function p (depending only on the dimension of X) such that for every $b \in X$ and every k ,*

$$\# \{ a \in X : c_{ab} = \pm z^k w^\ell \text{ for some } \ell \text{ and all } z \in \mathbb{D} \} \leq p(k)$$

and such that for every $a \in X$ and every k ,

$$\# \{ b \in X : c_{ab} = \pm z^k w^\ell \text{ for some } \ell \text{ and all } z \in \mathbb{D} \} \leq p(k).$$

Proof. Fix $b \in X$. Let $x = u_0, \dots, u_m = b$ be a geodesic edge-path from x to b . Let C be a cube containing the vertex b in which meet all the hyperplanes that are adjacent to b and that separate b from y . Let $b = v_0, \dots, v_n = y$ be a geodesic edge-path from b to y composed of a path v_0, \dots, v_i consisting of vertices in C , followed by a path v_{i+1}, \dots, v_n which does not cross any hyperplane adjacent to $b = v_0$ (see Proposition 2.9). By the cocycle property,

$$c_z(x, y) = c_z(u_0, u_1) \cdots c_z(u_{m-1}, u_m) c_z(v_0, v_1) \cdots c_z(v_{n-1}, v_n).$$

Furthermore $c_z(v_j, v_{j+1})\delta_b = \delta_b$ for all $j \geq i$ since b is not adjacent to the hyperplane separating v_j from v_{j+1} , and therefore $c_z(x, y)\delta_b = c_z(x, v_i)\delta_b$. Now, because b and v_i are vertices of the same cube C , it follows from Proposition 4.2 and Lemma 2.6 that there is another vertex c of C such that

$$\{ a \in X : c_{ab} \neq 0 \text{ for some } z \in \mathbb{D} \} \subseteq \mathfrak{C}(x, c).$$

Note next that $d(b, c) \leq \dim(X)$, so that if $d(a, b) = k$, then $d(a, c) \leq k + \dim(X)$. In addition, if $c_{ab} = \pm z^k w^\ell$, then we proved in Proposition 4.9 that $k = d(a, b)$, from which it follows that $d(a, c) \leq k + \dim(X)$. Therefore

$$\{ a \in X : c_{ab} = \pm z^k w^\ell \text{ for some } \ell \text{ and all } z \in \mathbb{D} \} \subseteq \mathfrak{C}(x, c) \cap B(c, k + \dim(X))$$

and Proposition 5.1 implies that for all k ,

$$\# \{ a \in X : c_{ab} = \pm z^k w^\ell \text{ for some } \ell \text{ and all } z \in \mathbb{D} \} \leq (k + d + 1)^d,$$

where $d = \dim(X)$.

To prove the second estimate, it suffices to note that $c_z(x, y) = c_{\bar{z}}(y, x)^*$, so that the ab -matrix coefficient of $c_z(x, y)$ is equal to the complex conjugate of the ba -matrix coefficient of $c_{\bar{z}}(y, x)$.

Thus the second estimate, using the same polynomial $p(k) = (k + d + 1)^d$, follows from the first. \square

Following the approach we took in Section 1, let us use the results of Section 4 to decompose the operator $c_z(x, y)$ as a linear combination

$$c_z(x, y) = \sum_{k \geq 0} z^k c_z^{(k)}(x, y),$$

in which the non-zero matrix coefficients of the operators $c_z^{(k)}(x, y)$ are all of the form $\pm w^\ell$, where $0 \leq \ell \leq 2 \dim(X)$. Note that this is actually a finite linear combination since, by Lemma 4.1, $c_z^{(k)}(x, y) = 0$ when $k \geq d(x, y)$.

5.3. Proposition. *For every compact subset $K \subseteq \mathbb{D}$,*

$$\sup \{ \|c_z(x, y)\| : x, y \in X, z \in K \} < \infty.$$

Proof. Since $|w|^2 \leq 2$, the matrix entries of $c_z^{(k)}(x, y)$ are all bounded in absolute value by $2^{\dim(X)}$. Furthermore there are at most $p(k)$ non-zero matrix entries in each row and column, where p is the polynomial function of Lemma 5.2. Therefore, as in Section 1,

$$\|c_z^{(k)}(x, y)\| \leq 2^{\dim(X)} p(k)$$

It follows that

$$\|c_z(x, y)\| \leq 2^{\dim(X)} \sum_{k=0}^{\infty} |z|^k p(k)$$

and this gives the result. \square

Having established that the cocycle $c_z(x, y)$ is uniformly bounded, the proof of our main theorem is complete:

5.4. Definition. A *CAT(0)-cubical group* is a group G which admits an action on a finite-dimensional CAT(0) cube complex in such a way that $d(gx, x)$ is a proper function on G for some (and hence any) vertex x .

5.5. Theorem. *If G is a CAT(0)-cubical group, then G is weakly amenable and has Cowling-Haagerup constant 1.* \square

APPENDIX A. A BOUND ON THE INTERSECTIONS OF INTERVALS WITH BALLS

We shall give a new proof, which may be of independent interest, of the following result of Chatterji and Ruane [CR05].

A.1. Theorem. *Let X be a CAT(0) cube complex and let x, y be vertices of X . For every $r \in \mathbb{N}$ the cardinality of the set $\mathcal{C}(x, y) \cap B(x, r)$ is bounded by $(r + 1)^d$, where d is the dimension of X .*

The proof relies on the following proposition.

A.2. Proposition. *Let X be a CAT(0) cube complex and let x and y be vertices of X . Let $H \in \mathfrak{H}(x, y)$ and suppose that x is adjacent to H . There exists a second vertex v adjacent to H and on the same side of H as x for which*

$$\mathfrak{C}(x, y) = \mathfrak{C}(x, v) \cup \mathfrak{C}(x^{\text{op}}, y) \quad (\text{disjoint union}).$$

A.3. Remark. In fact we only require the forward inclusion for Theorem A.1.

We shall also use the fact that a hyperplane H in a CAT(0) cube complex may be given a natural CAT(0) cube complex structure in its own right. The set underlying H can be taken to be the set of vertices in ∂H_+ , for any fixed orientation of H . The cubes in H are exactly the subsets of H that are cubes in X . Distance and convex hulls may be computed in H or in the ambient complex X . See [Sag95].

Proof of Theorem A.1, assuming Proposition A.2. The proof is by double induction on the dimension d of X and the radius r . The base of the induction comprises two cases. The result is obvious in the case of arbitrary d and $r = 0$ since $B(x, 0)$ contains only x . It is also obvious in the case of arbitrary r and $d = 0$.

For the induction step, given $\mathbf{d} > 0$ and $\mathbf{r} > 0$, assume the result for all complexes of dimension less than \mathbf{d} , no matter what the value of r , and for all complexes X of dimension $d = \mathbf{d}$ and all balls in X of radius $r < \mathbf{r}$. Now assume that $\dim(X) = \mathbf{d}$ and let $r = \mathbf{r}$. Let $x, y \in X$ and let H be a hyperplane adjacent to x that separates x from y . It follows from the proposition that

$$\mathfrak{C}(x, y) \cap B(x, r) \subseteq (\mathfrak{C}(x^{\text{op}}, y) \cap B(x^{\text{op}}, r - 1)) \cup (\mathfrak{C}(x, v) \cap B(x, r))$$

(note that if $u \in \mathfrak{C}(x^{\text{op}}, y)$ then $d(x^{\text{op}}, u) = d(x, u) - 1$ since H separates x from u but not x^{op} from u). The induction hypothesis implies that

$$\# (\mathfrak{C}(x^{\text{op}}, y) \cap B(x^{\text{op}}, r - 1)) \leq r^{\mathbf{d}}.$$

Since $\mathfrak{C}(x, v) \subseteq H$ and since the dimension of H is less than \mathbf{d} , the induction hypothesis also implies that

$$\# (\mathfrak{C}(x, v) \cap B(x, r)) \leq (r + 1)^{\mathbf{d}-1}.$$

Combining these estimates, we conclude

$$\# (\mathfrak{C}(x, y) \cap B(x, r)) \leq r^{\mathbf{d}} + (r + 1)^{\mathbf{d}-1} \leq (r + 1)^{\mathbf{d}},$$

as required. □

The remainder of the appendix is devoted to proving Proposition A.2. Let x and y be vertices of X and let $H \in \mathfrak{H}(x, y)$ be such that $x \in \partial H$. Orient H by the requirement that $x \in H_+$. Let v be an element of the finite set $\mathfrak{C}(x, y) \cap H_+$ of maximal distance from x (or equivalently of minimal distance to y). We shall show that v has the properties stated in the proposition.

A.4. Lemma. *Every vertex in $\mathfrak{C}(x, y) \cap H_+$ is adjacent to H . In particular, v is adjacent to H .*

Proof. Let $u \in \mathcal{C}(x, y) \cap H_+$ and suppose for the sake of a contradiction that u is not adjacent to H . Let u_0, \dots, u_n be a geodesic from x to y containing u . There is a first vertex u_j on this geodesic that is not adjacent to H . Then of course u_{j-1} is adjacent to H . Let K be the hyperplane separating u_{j-1} from u_j . Both H and K separate u_{j-1} from y . It follows that there is a square containing u_{j-1} as a vertex in which H and K intersect, and as in the proof of Proposition 4.9, it follows that u_j is adjacent to H . Contradiction. \square

A.5. Lemma. *If $K \in \mathfrak{H}(v, y)$ and if K is adjacent to v , then $K = H$.*

Proof. If $K \neq H$, then since both H and K separate v from y , and since both are adjacent to v , the two hyperplanes intersect in a square containing v as a vertex. The vertex v^{op} adjacent to v across K is therefore adjacent to H . It is also in the interval from x to y and further away (by one) from x than v . This contradicts the definition of v . \square

In the next lemma we require a small amount of the theory of *normal cube paths*. See [NR98a] for further information, especially the remark following Proposition 3.3.

A.6. Lemma. *No hyperplane $K \neq H$ separating v and y can intersect H .*

Proof. Assume for the sake of a contradiction that such a hyperplane K exists. There is then a hyperplane K other than H that separates v from y and which has the property that if K intersects the normal cube C , then no hyperplane (other than H itself) intersecting a normal cube prior to C on the normal cube path from v to y intersects H . Observe that the cube C cannot be the first cube in the normal cube path from v to y since by the previous lemma K cannot be adjacent to v . Let K_1, \dots, K_d be the hyperplanes spanning the normal cube C' immediately preceding C . A simple separation argument based on Lemma 2.10 shows that $K \cap K_i$ is nonempty for each i . Thus, the hyperplanes K, K_1, \dots, K_d intersect pairwise, and each is adjacent to the (unique) vertex w in $C' \cap C$. By Proposition 2.7 each pair intersects in a square having w as a vertex. But, the link of w is a flag complex (see [BH99, Thm. II.5.20]), so that these hyperplanes intersect in a cube having w as a vertex. This contradicts the definition of normal cube path according to which $\text{St}(C') \cap C = \{w\}$. \square

Proof of Proposition A.2. We shall show that

$$(A.1) \quad \mathcal{C}(x, y) \cap H_- = \mathcal{C}(x^{\text{op}}, y)$$

and

$$(A.2) \quad \mathcal{C}(x, y) \cap H_+ = \mathcal{C}(x, v).$$

Let $u \in \mathcal{C}(x, y) \cap H_-$. Then $d(x, u) + d(u, y) = d(x, y)$. Since $u \in H_-$ it follows that $d(x^{\text{op}}, u) = d(x, u) - 1$. Therefore

$$d(x^{\text{op}}, u) + d(u, y) = d(x, u) - 1 + d(u, y) = d(x, y) - 1 = d(x^{\text{op}}, y)$$

so that $u \in \mathcal{C}(x^{\text{op}}, y)$. The other inclusion in (A.1) is obvious.

It follows immediately from [Sag95, Thm. 4.13] that $\mathcal{C}(x, v) \subseteq H_+$. Using Corollary 2.2, if $w \in \mathcal{C}(x, v)$ then $\mathfrak{H}(x, w) \subseteq \mathfrak{H}(x, v) \subseteq \mathfrak{H}(x, y)$ so that $w \in \mathcal{C}(x, y)$.

Finally, let $w \in \mathcal{C}(x, y) \cap H_+$. Then by Corollary 2.2,

$$\mathfrak{H}(x, w) \subseteq \mathfrak{H}(x, y) = \mathfrak{H}(x, v) \cup \mathfrak{H}(v, y) \quad (\text{disjoint union}).$$

We want to show that $\mathfrak{H}(x, w) \subseteq \mathfrak{H}(x, v)$, or equivalently that $\mathfrak{H}(x, w) \cap \mathfrak{H}(v, y)$ is empty. This is a separation argument. Indeed, suppose $K \in \mathfrak{H}(v, y)$. If $K = H$ then $K \notin \mathfrak{H}(x, w)$ by assumption. If $K \neq H$ then K and H are parallel by Lemma A.6. Further, K is contained entirely in H_- , since $K \in \mathfrak{H}(v^{\text{op}}, y)$ and both v^{op} and $y \in H_-$. Now, a geodesic path from x to w is completely contained in H_+ , so cannot cross H , and therefore cannot cross K . That is, $K \notin \mathfrak{H}(x, w)$. \square

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UNIVERSITY OF HAWAI'I, MĀNOA, DEPARTMENT OF MATHEMATICS, 2565 MCCARTHY MALL, HONOLULU, HI 96822-2273

E-mail address: erik@math.hawaii.edu

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802

E-mail address: higson@math.psu.edu