NEW C*-COMPLETIONS OF DISCRETE GROUPS AND RELATED SPACES

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Abstract. Let Γ be a discrete group. To every ideal in $\ell^\infty(\Gamma)$ we associate a C*-algebra completion of the group ring that encapsulates the unitary representations with matrix coefficients belonging to the ideal. The general framework we develop unifies some classical results and leads to new insights. For example, we give the first C*-algebraic characterization of a-T-menability; a new characterization of property (T); new examples of "exotic" quantum groups; and, after extending our construction to transformation groupoids, we improve and simplify a recent result of Douglas and Nowak [8].

1. Introduction

Since their introduction by von Neumann, amenable groups have played an important role in many areas of mathematics. They have been studied from a variety of perspectives and in many different contexts, and a vast literature is now devoted to them. More recently, the concept of a measurable amenable action of a (non-amenable) group was introduced by Zimmer. Both this and a topological version have been subsequently developed by many authors. An elementary connection between these theories is the fact that every action of an amenable group is an amenable action. Less obvious, but equally well-known, is that if a group acts amenably on a compact space fixing a probability measure then the group itself is amenable.

This last fact is the launching point of a recent paper by Douglas and Nowak [8], in which, among other things, they introduce conditions on an amenable action sufficient to guarantee that the group acting is a-T-menable – in other words, that it admits a metrically proper, affine isometric action on a Hilbert space. An amenable group is a-T-menable, so that one may imagine hypotheses involving existence of a quasi-invariant measure together with conditions on the associated Radon-Nikodym cocycle. Precisely, suppose a discrete group Γ acts amenably on the compact Hausdorff topological space X, and that μ is a probability measure on X which is quasi-invariant for the action. Define upper and lower envelopes of the Radon-Nikodym cocycle by

$$\overline{\rho}(x) = \sup_{s\in G} \frac{ds^*\mu}{d\mu}(x), \quad \text{and} \quad \underline{\rho}(x) = \inf_{s\in G} \frac{ds^*\mu}{d\mu}(x);$$

The first named author was partially supported by DMS-0856197. The second named author was partially supported by DMS-0349367.

here, $s^*\mu$ is the translate of the measure μ by the group element s, and $ds^*\mu/d\mu$ is the Radon-Nikodym derivative. Douglas and Nowak show that if $\overline{\rho}$ is integrable, or if $\underline{\rho}$ is nonzero, then the group Γ is a-T-menable. They ask whether amenability of Γ follows from either of these conditions. In this note, we shall prove that this is indeed the case. See Corollary 5.11 and surrounding discussion.

Our initial result led us to the following question: if one wishes to conclude a-T-menability of Γ, what are the appropriate hypotheses? To answer this question, we introduce appropriate completions of the group ring of Γ, and of the convolution algebra $C_c(X \rtimes G)$ in the case of an action. Precisely, for every algebraic ideal in $\ell^\infty(\Gamma)$ we associate a completion – for $\ell^\infty(\Gamma)$ we recover the full C*-algebra, for $c_c(\Gamma)$ we recover the reduced C*-algebra, and for $c_0(\Gamma)$ we obtain new C*-algebras well-adapted to the study of a-T-menability and a-T-menable actions. Our results in this context are summarized:

Theorem. *Let Γ be a discrete group acting on a compact Hausdorff topological space X. Let $C^*_{c_0}(\Gamma)$ and $C^*_{c_0}(X \rtimes \Gamma)$ be the completions with respect to the ideal $c_0(\Gamma)$. We have:*

(1) *Γ is a-T-menable if and only if $C^*(\Gamma) = C^*_{c_0}(\Gamma)$;*
(2) *if the action of Γ on X is a-T-menable (see Definition 5.5) then $C^*(X \rtimes \Gamma) = C^*_{c_0}(X \rtimes \Gamma)$.*

Further, under the hypotheses of Douglas and Nowak, Γ is a-T-menable if and only if its action on X is a-T-menable.

Apart from this theorem, and ancillary related results, we develop some general aspects of our ideal completions. We study when an ideal completion recovers the full or reduced group C*-algebra and give examples when it is neither – this gives rise to 'exotic' compact quantum groups. We recover a standard characterization of amenability – equality of the full and reduced group C*-algebras – we obtain the characterization of a-T-menability stated above, and we characterize Property (T) in terms of ideal completions.

Acknowledgement. The first author thanks the math department at the University of Hawai'i for embodying the aloha spirit during the sabbatical year when this work was carried out. He also thanks Yehuda Shalom and Rufus Willett for helpful remarks and suggestions, respectively. Both authors thank Jesse Peterson for sharing his insights. Finally, we thank Thibault Pillon for reading the paper in Alain Valette's seminar, and pointing out an error in our original proof of Proposition 3.6

2. Ideals and C*-completions

Throughout, Γ will denote a (countable) discrete group and $D \triangleleft \ell^\infty(\Gamma)$ will be an *algebraic* (not necessarily norm-closed) two-sided ideal. If $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$ is a unitary representation and vectors ξ, $\eta \in \mathcal{H}$ are given, the ℓ^∞-function

$$\pi_{\xi,\eta}(s) := \langle \pi_s(\xi), \eta \rangle$$

is a *matrix coefficient* (*function*) of π. The map associating to a pair of vectors their matrix coefficient function is sesquilinear; concretely, given finitely many vectors v_i, $w_j \in \mathcal{H}$, if we set $\xi = \sum \alpha_i v_i$ and $\eta = \sum_j \beta_j w_j$ then we have

$$\pi_{\xi,\eta} = \sum_{i,j} \alpha_i \bar{\beta}_j \pi_{v_i,w_j}.$$

In particular, if a linear subspace of $\ell^\infty(\Gamma)$ contains the π_{v_i,w_j} then it contains $\pi_{\xi,\eta}$ as well.

Definition 2.1. Let $D \triangleleft \ell^\infty(\Gamma)$ be an ideal. A unitary representation $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$ is a *D-representation* if there exists a dense linear subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\pi_{\xi,\eta} \in D$ for all ξ, $\eta \in \mathcal{H}_0$.

Definition 2.2. Let $D \triangleleft \ell^\infty(\Gamma)$ be an ideal. Define a C*-norm on the group ring $\mathbb{C}[\Gamma]$ by

$$\|x\|_D := \sup\{\, \|\pi(x)\| : \pi \text{ is a } D\text{-representation}\};$$

let $C^*_D(\Gamma)$ denote the completion of $\mathbb{C}[\Gamma]$ with respect to $\|\cdot\|_D$.

We shall refer to the C*-algebra $C^*_D(\Gamma)$, and its generalizations defined below, as *ideal completions*; these will be our primary objects of study.

Evidently, $C^*_D(\Gamma)$ has the universal property that every D-representation of Γ extends uniquely to $C^*_D(\Gamma)$. We shall refer to such representations as *D-representations* of $C^*_D(\Gamma)$.

By virtue of its universal property, the full group C*-algebra of Γ surjects onto every ideal completion. For some D the ideal completion $C^*_D(\Gamma)$ does not contain the group ring – it may even be the zero C^*-algebra! – so is not strictly speaking a 'completion'. However, if D contains the ideal $c_c(\Gamma)$ of finitely supported functions then $C^*_D(\Gamma)$ is indeed a completion of the group ring – this follows because the regular representation of Γ, being a c_c-representation, extends to $C^*_D(\Gamma)$.

Remark 2.3. If D is a *closed* ideal, then *every* matrix coefficient of a D-representation belongs to D (that is, not just those associated to the dense subspace $\mathcal{H}_0$). For example, this is the case for the ideal $c_0(\Gamma)$ of functions vanishing at infinity.

Remark 2.4 (Tensor products). The tensor product of a D-representation and an arbitrary representation is again a D-representation. Suppose $\pi\colon \Gamma \to B(\mathcal{H})$ is a D-representation and $\sigma\colon \Gamma \to B(\mathcal{K})$ is arbitrary. For $v_i \in \mathcal{H}_0$ and $w_i \in \mathcal{K}$ we have

$$(\pi \otimes \sigma)_{v_1\otimes w_1, v_2\otimes w_2} = \pi_{v_1,v_2}\sigma_{w_1,w_2} \in D,$$

since D is an ideal; further, such simple tensors have dense span in $\mathcal{H} \otimes \mathcal{K}$.

Remark 2.5 (Direct sums). An arbitrary direct sum of D-representations is again a D-representation. This follows since in the definition we only require a dense subspace.

As a consequence, $C^*_D(\Gamma)$ has a *faithful* D-representation. Indeed, for each element $x \in C^*_D(\Gamma)$ there is a D-representation π such that $\pi(x) \neq 0$. Taking direct sums one easily constructs a faithful D-representation of $C^*_D(\Gamma)$.

Definition 2.6. An ideal $D \triangleleft \ell^\infty(\Gamma)$ is *translation invariant* if it is invariant under both the left and right translation actions of Γ on $\ell^\infty(\Gamma)$.

Every nonzero, translation invariant ideal in $\ell^\infty(\Gamma)$ contains the ideal $c_c(\Gamma)$. It follows that the ideal completion with respect to a translation invariant ideal surjects onto the reduced group C*-algebra of Γ.

Remark 2.7 (Cyclic representations). Let D be a translation invariant ideal. If v is a cyclic vector for a representation $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$ and $\pi_{v,v} \in D$, then π is a D-representation. Indeed, a computation confirms that if $\xi = \pi_{g_1}(v)$ and $\eta = \pi_{g_2}(v)$, then

$$\pi_{\xi,\eta}(s) = \pi_{v,v}(g_2^{-1} s g_1),$$

so that also $\pi_{\xi,\eta} \in D$. Setting $\mathcal{H}_0 = \operatorname{span}\{\pi_s(v) : s \in \Gamma\}$ we see that $\mathcal{H}_0$ is dense in $\mathcal{H}$ and that the matrix coefficients coming from vectors in $\mathcal{H}_0$ belong to D.

Proposition 2.8. *Let $\phi : \Gamma_1 \to \Gamma_2$ be a group homomorphism and let $D_i \lhd \ell^\infty(\Gamma_i)$ be ideals satisfying the following condition: if $f \in D_2$ then $f \circ \phi \in D_1$. Then ϕ extends to a* C*-*homomorphism $C^*_{D_1}(\Gamma_1) \to C^*_{D_2}(\Gamma_2)$.*

Proof. Apply the following simple observation to a faithful D_2-representation of Γ_2: under the stated hypotheses, if π is a D_2-representation of Γ_2, then $\pi \circ \phi$ is a D_1-representation of Γ_1; in particular it extends to $C^*_{D_1}(\Gamma_1)$. $\square$

Corollary 2.9. *The following assertions hold.*

1. *Suppose $D_i \lhd \ell^\infty(\Gamma)$ are ideals and $D_2 \subset D_1$; there is a quotient map $C^*_{D_1}(\Gamma) \to C^*_{D_2}(\Gamma)$ (extending the identity map on the group ring).*
2. *Suppose $\Lambda \subset \Gamma$ is a normal subgroup, $D_2 \lhd \ell^\infty(\Gamma/\Lambda)$ is an ideal and $D_1 \lhd \ell^\infty(\Gamma)$ is an ideal containing the image of D_2 under the inclusion $\ell^\infty(\Gamma/\Lambda) \subset \ell^\infty(\Gamma)$; there is a surjection (extending the homomorphism ϕ)*

$$C^*_{D_1}(\Gamma) \to C^*_{D_2}(\Gamma/\Lambda).$$

Proof. Both statements are immediate from the proposition. The first is also equivalent to the inequality, $\|\cdot\|_{D_1} \leq \|\cdot\|_{D_2}$, which is immediate from the definitions. $\square$

We close this introductory section by looking at several basic examples. Our first example is trivial, since both algebras in question satisfy the same universal property.

Proposition 2.10. *For every discrete group Γ, the completion with respect to the ideal $\ell^\infty(\Gamma)$ is the universal (or full) group* C*-*algebra: $C^*_{\ell^\infty}(\Gamma) = C^*(\Gamma)$.* $\square$

Proposition 2.11. *For every discrete group Γ and every $p \in [1,2]$ the completion with respect to the ideal $\ell^p(\Gamma)$ is the reduced group* C*-*algebra: $C^*_{\ell^p}(\Gamma) = C^*_r(\Gamma)$.*

Proof. This follows from the Cowling-Haagerup-Howe Theorem (cf. [7]): if $\pi\colon \Gamma \to B(\mathcal{H})$ has a cyclic vector $v \in \mathcal{H}$ and $\pi_{v,v} \in \ell^2(\Gamma)$, then π is weakly contained in the regular representation.[1] Indeed, fix a nonzero $x \in C^*_{\ell^p}(\Gamma)$. We can find a cyclic $\ell^p(\Gamma)$-representation

[1]Actually, for the Cowling-Haagerup-Howe Theorem it suffices to have $\pi_{v,v} \in \ell^{2+\varepsilon}(\Gamma)$ for all $\varepsilon > 0$. Thus, the proposition generalizes to the ideal $D := \cap_{\varepsilon>0}\ell^{2+\varepsilon}(\Gamma)$, with exactly the same proof.

π such that $\pi(x) \neq 0$ – simply restrict a faithful $\ell^p(\Gamma)$-representation to an appropriate cyclic subspace. Since π is weakly contained in the regular representation, x cannot be in the kernel of the map $C^*_{\ell^p}(\Gamma) \to C^*_r(\Gamma)$. □

It follows from functoriality (part (1) of Corollary 2.9) that if a translation invariant ideal D is contained in $\ell^p(\Gamma)$ for some $p \in [1,2]$ then $C^*_D(\Gamma) = C^*_r(\Gamma)$. This applies, in particular, to the ideal of finitely supported functions. In contrast, the ideals $\ell^p(\Gamma)$ for finite p give rise to the universal group C*-algebra only if Γ is amenable.

Proposition 2.12. *If there exists $p \in [1,\infty)$ for which $C^*(\Gamma) = C^*_{\ell^p}(\Gamma)$, then Γ is amenable.*

In the proof, and at a number of places below, we shall use the notion of a positive definite function: recall that $h\colon \Gamma \to \mathbb{C}$ is *positive definite* if for every $s_1,\dots,s_n \in \Gamma$, the matrix $[h(s_i s_j^{-1})]_{i,j} \in \mathbb{M}_n(\mathbb{C})$ is positive (semidefinite).

Proof. If $C^*(\Gamma) = C^*_{\ell^p}(\Gamma)$, then $C^*(\Gamma)$ admits a *faithful* $\ell^p(\Gamma)$-representation π and, taking an infinite direct sum if necessary, we may assume $\pi(C^*(\Gamma))$ contains no compact operators. In this case, Glimm's lemma implies that π weakly contains the trivial representation. Thus, let v_n be unit vectors such that $\|\pi_s(v_n) - v_n\| \to 0$ for all $s \in \Gamma$. Approximating the v_n's with vectors having associated matrix coefficients in $\ell^p(\Gamma)$, we may assume $\pi_{v_n,v_n} \in \ell^p(\Gamma)$ for all $n \in \mathbb{N}$. Since π_{v_n,v_n} are positive definite functions tending pointwise to one, we conclude that Γ is amenable. □

Remark 2.13. In the previous proof we have used the following elementary fact: if there exist positive definite functions $h_n \in \ell^p(\Gamma)$ for which $h_n \to 1$ pointwise, then Γ is amenable. Lacking a reference, we provide the following argument. For k larger than p the functions h_n^k are positive definite, converge pointwise to one, and belong to $\ell^1(\Gamma) \subset C^*_r(\Gamma)$. To get finitely supported functions with similar properties, consider $f_n \in \mathbb{C}(\Gamma)$ which approximate the square roots of the h_n^k in the norm of $C^*_r(\Gamma)$, so that h_n^k is approximated by the finitely supported positive definite function $f_n^* * f_n$.

3. Positive definite functions and the Haagerup property

Though very simple, the proof of Proposition 2.12 suggests a general result. We begin with a lemma isolating the role of translation invariance.

Lemma 3.1. *Suppose $D \lhd \ell^\infty(\Gamma)$ is a translation invariant ideal and $h \in D$ is positive definite. The GNS representation corresponding to h is a D-representation.*

Proof. Immediate from Remark 2.7: if π is the GNS representation associated to h, and $v \in \mathcal{H}$ is the canonical cyclic vector, then $\pi_{v,v} = h \in D$. □

Theorem 3.2. *Let $D \lhd \ell^\infty(\Gamma)$ be a translation invariant ideal. We have that $C^*(\Gamma) = C^*_D(\Gamma)$ if and only if there exist positive definite functions $h_n \in D$ converging pointwise to the constant function 1.*

Proof. First assume that the canonical map $C^*(\Gamma) \to C^*_D(\Gamma)$ is an isomorphism. Replacing $\ell^p(\Gamma)$ with D in the proof of Proposition 2.12, we see how to construct the desired positive definite functions.

For the converse, suppose $h_n \in D$ are positive definite functions such that $h_n(s) \to 1$. To prove that $C^*(\Gamma) = C^*_D(\Gamma)$, it suffices to observe that vector states coming from D-representations are weak-$*$ dense in the state space of $C^*(\Gamma)$, since this implies that the map $C^*(\Gamma) \to C^*_D(\Gamma)$ has a trivial kernel. So let φ be a state on $C^*(\Gamma)$. Then the formula

$$\varphi_n\left(\sum_{s\in\Gamma}\alpha_s s\right) := \sum_{s\in\Gamma}\alpha_s h_n(s)\varphi(s)$$

determines a state on $C^*(\Gamma)$ – it is the composition of φ and the completely positive Schur multiplier $C^*(\Gamma) \to C^*(\Gamma)$ associated to h_n, cf. [5]. Since the norms of the φ_n are uniformly bounded, we have that $\varphi_n \to \varphi$ in the weak-$*$ topology. Also, it's clear that $\varphi_n|_\Gamma \in D$ since it is the product of h_n and $\varphi|_\Gamma$. So the previous lemma implies the GNS representations associated to the φ_n are D-representations, concluding the proof. □

It has been open for some time whether the Haagerup property ($\equiv$ a-T-menability, see [6]) admits a C*-algebraic characterization. The previous theorem easily implies such a characterization, which is perfectly analogous to a well-known fact about amenable groups. To see the parallel, we isolate two more canonical ideal completions.

Definition 3.3. Let $C^*_{c_c}(\Gamma)$ denote the ideal completion associated to the ideal $c_c(\Gamma)$ of finitely supported functions; let $C^*_{c_0}(\Gamma)$ denote the ideal completion associated to $c_0(\Gamma)$, the functions vanishing at infinity.

Recall that Γ is amenable if there exist positive definite functions in $c_c(\Gamma)$ converging pointwise to one; similarly Γ has the Haagerup property if there exist positive definite functions in $c_0(\Gamma)$ converging pointwise to one. Since both $c_c(\Gamma)$ and $c_0(\Gamma)$ are translation invariant ideals, our next result follows immediately from Theorem 3.2. Having already observed that $C^*_{c_c}(\Gamma) = C^*_r(\Gamma)$, the first statement is classical. The second statement is closely related to the following fact: Γ has the Haagerup property if and only if it admits a c_0-representation weakly containing the trivial representation [6].

Corollary 3.4. *For a discrete group Γ we have: Γ is amenable if and only if $C^*(\Gamma) = C^*_{c_c}(\Gamma)$; Γ has the Haagerup property if and only if $C^*(\Gamma) = C^*_{c_0}(\Gamma)$.* □

Since $C^*_{c_0}(\Gamma)$ admits a faithful c_0-representation, we have an analogue of the fact that every representation of an amenable group is weakly contained in the left regular representation.

Corollary 3.5. *If Γ has the Haagerup property, then every unitary representation is weakly contained in a c_0-representation.*[2] □

[2]This can also be deduced from the existence of a c_0-representation weakly containing the trivial one.

Jesse Peterson asked if Property (T) (see [2]) can be characterized in this context, and suggested the following proposition. Thibault Pillon pointed out an error in our original proof and, a short time after we had arrived at the argument presented here, supplied a correct proof which is slightly less elementary and arguably slicker that our own.

Proposition 3.6. *A discrete group* Γ *has Property* (T) *precisely when the following condition holds:* $D = \ell^\infty(\Gamma)$ *is the only translation invariant ideal* D *for which* $C^*(\Gamma) = C^*_D(\Gamma)$.

Proof. First, suppose that Γ has Property (T) and that D is a translation invariant ideal for which $C^*_D(\Gamma) = C^*(\Gamma)$. We must show that $D = \ell^\infty(\Gamma)$. But, by Theorem 3.2 there exist positive definite functions $h_n \in D$ converging pointwise to one. Since Γ has Property (T) they converge uniformly to one. Thus, some h_n is bounded away from zero, and so is invertible in $\ell^\infty(D)$.

Conversely, assume Γ does *not* have Property (T). We shall construct a *proper* translation invariant ideal D for which $C^*_D(\Gamma) = C^*(\Gamma)$. To start, let $\pi\colon \Gamma \to B(\mathcal{H})$ be a unitary representation and $b\colon \Gamma \to \mathcal{H}$ be an *unbounded* cocycle for π – existence of these is implied by (indeed equivalent to) the fact that Γ does *not* have Property (T) (cf. [2]). The functions $h_n \in \ell^\infty(\Gamma)$ defined by

$$h_n(t) = \exp(-\|b(t)\|^2/n)$$

are positive definite (cf. [2]) and converge pointwise to one. Thus, by Theorem 3.2, it suffices to show that the translation invariant ideal they generate, D, is proper. Because an ideal in $\ell^\infty(\Gamma)$ is proper if and only if it does not contain any invertible elements it suffices to show that no function in this ideal is invertible.

Now, an element of D necessarily has the form

$$t \longmapsto f_1 h_{n_1}(s_1^{-1} t u_1) + \ldots + f_k h_{n_k}(s_k^{-1} t u_k),$$

for some $n_1, \ldots, n_k \in \mathbb{N}$, $f_1, \ldots, f_k \in \ell^\infty(\Gamma)$, and $s_1, \ldots, s_k, u_1, \ldots, u_k \in \Gamma$. To see that this function is not invertible, we show that for every $\varepsilon > 0$ there exists a group element $t \in \Gamma$ such that

$$|f_1 h_{n_1}(s_1^{-1} t u_1) + \ldots + f_k h_{n_k}(s_k^{-1} t u_k)| < \varepsilon.$$

Recall that b satisfies the following *cocycle identity* with respect to π: for every $x, y \in \Gamma$,

$$b(xy) = \pi(x)b(y) + b(x).$$

Given s, $\tilde{t} \in \Gamma$ apply this with $x = s$, $y = s^{-1}\tilde{t}$ to see that $\pi(s)b(s^{-1}\tilde{t}) = b(\tilde{t}) - b(s)$; if $\tilde{t} = tu$ for some t, $u \in \Gamma$, we apply the identity again to see that

$$\pi(s)b(s^{-1}tu) = \pi(t)b(u) + b(t) - b(s).$$

Since b is unbounded, it follows – and this is the important part for us – that if $s_1, \ldots, s_k$, and $u_1, \ldots, u_k \in \Gamma$ are given and $C > 0$ is arbitrary, there exists an element $t \in \Gamma$ such that

$$\|b(s_i^{-1} t u_i)\|^2 > C$$

for all $i \in \{1, \ldots, k\}$. From here, the desired assertion is routine. ☐

4. Quantum groups

Recall that a *compact quantum group* is a pair (A, Δ) where A is a unital C*-algebra and $\Delta\colon A \to A \otimes A$ is a unital $*$-homomorphism satisfying the following two conditions:

(1) $(\Delta \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \Delta)\Delta$, and
(2) $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are dense subspaces of $A \otimes A$.[3]

The map Δ is the *co-multiplication* and the first property is called *co-associativity*.

Discrete groups provided an early source of examples of quantum groups. Indeed, the assignment $\Delta(s) = s \otimes s$ on group elements determines a co-associative map

$$\Delta\colon \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma], \tag{1}$$

and one can check that $\Delta(\mathbb{C}[\Gamma])(\mathbb{C}[\Gamma] \otimes 1) = \Delta(\mathbb{C}[\Gamma])(1 \otimes \mathbb{C}[\Gamma]) = \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma]$. Thus, if A is a C*-algebra containing $\mathbb{C}[\Gamma]$ as a dense subalgebra and for which Δ can be extended continuously to a map $A \to A \otimes A$, then A is a compact quantum group. Every discrete group gives rise to two canonical compact quantum groups: the universal property ensures that $A = C^*(\Gamma)$ is a compact quantum group whereas Fell's absorption principle implies that $A = C^*_r(\Gamma)$ is a compact quantum group. Anything between these extremes is considered 'exotic' (cf. [11]). The purpose of this section is to provide examples of such exotic compact quantum groups.

Proposition 4.1. *For every ideal $D \triangleleft \ell^\infty(\Gamma)$, the ideal completion $C^*_D(\Gamma)$ is a compact quantum group.*

Proof. We shall show that the map Δ in (1) extends continuously to $C^*_D(\Gamma) \to C^*_D(\Gamma) \otimes C^*_D(\Gamma)$. To this end, fix a faithful D-representation $C^*_D(\Gamma) \subset B(\mathcal{H})$. By Remark 2.9, we can regard the composite

$$\Delta\colon \mathbb{C}[\Gamma] \to C^*_D(\Gamma) \otimes C^*_D(\Gamma) \subset B(\mathcal{H} \otimes \mathcal{H})$$

as a D-representation and the universal property ensures that Δ extends to $C^*_D(\Gamma)$. $\square$

Thus our task is to provide examples of groups Γ and ideals $D \triangleleft \ell^\infty(\Gamma)$ for which $C^*(\Gamma) \neq C^*_D(\Gamma) \neq C^*_r(\Gamma)$. Though a bit ad hoc, our first examples are easy to handle.

Proposition 4.2. *Suppose that Γ has Property (T) and that $\Lambda \triangleleft \Gamma$ is a non-amenable normal subgroup of infinite index. Suppose $D_1 \triangleleft \ell^\infty(\Gamma)$ and $D_2 \triangleleft \ell^\infty(\Gamma/\Lambda)$ are ideals satisfying the following conditions:*

(1) *D_1 is proper and translation invariant;*
(2) *D_2 is translation invariant;*
(3) *D_1 contains the image of D_2 under the inclusion $\ell^\infty(\Gamma/\Lambda) \subset \ell^\infty(\Gamma)$.*

Then $C^*_{D_1}(\Gamma)$ is an exotic compact quantum group.

[3] All tensor products in this definition are spatial (cf. [5]).

Proof. We must show that $C^*(\Gamma) \neq C^*_{D_1}(\Gamma) \neq C^*_r(\Gamma)$. Since D_1 is proper and translation invariant, and Γ has Property (T) the first inequality follows from Proposition 3.6.

To prove the second inequality, suppose to the contrary that $C^*_{D_1}(\Gamma) = C^*_r(\Gamma)$. Applying (2) of Corollary 2.9 we obtain a $*$-homomorphism

$$C^*_r(\Gamma) = C^*_{D_1}(\Gamma) \to C^*_{D_2}(\Gamma/\Lambda) \to C^*_r(\Gamma/\Lambda)$$

extending the homomorphism $\Gamma \to \Gamma/\Lambda$. It follows that Λ is amenable – the composite $C^*_r(\Lambda) \subset C^*_r(\Gamma) \to C^*_r(\Gamma/\Lambda)$ defines a character of $C^*_r(\Lambda)$. $\square$

Remark 4.3. While the hypotheses of the previous proposition may seem a bit contrived, examples are plentiful. An extension of Property (T) groups will again have Property (T) [2, Proposition 1.7.6]. As for the ideals, taking D_1 to be the ideal generated by the image of D_2 in $\ell^\infty(\Gamma)$ we have: if D_2 is translation invariant, then so is D_1; if $D_2 \subset c_0(\Gamma/\Lambda)$, then D_1 is proper.

Free groups also provide natural examples. We thank Rufus Willett for suggesting the proof of the following result.

Proposition 4.4. *Let $\mathbb{F}$ be a free group on two or more generators. There exists a $p \in (2,\infty)$ such that $C^*(\mathbb{F}) \neq C^*_{\ell^p}(\mathbb{F}) \neq C^*_r(\mathbb{F})$.*

Proof. Since $\mathbb{F}$ is not amenable, Proposition 2.12 implies that $C^*(\mathbb{F}) \neq C^*_{\ell^p}(\mathbb{F})$ for all finite p. We must find some p such that $C^*_{\ell^p}(\mathbb{F}) \neq C^*_r(\mathbb{F})$.

Let $S \subset \mathbb{F}$ be the standard symmetric generating set and let $|\cdot|$ denote the corresponding word length. A seminal result, first proved by Haagerup [10], states that for every $n \in \mathbb{N}$,

$$h_n(s) := e^{-|s|/n}$$

is positive definite. Clearly $h_n \to 1$ pointwise. Fixing n, we have $h_n \in \ell^{p_n}(\Gamma)$ for sufficiently large p_n; indeed if p_n is chosen so that $|S| < e^{p_n/n}$, or equivalently $|S|e^{-p_n/n} < 1$, then

$$\sum_{s\in\Gamma}(e^{-|s|/n})^{p_n} = \sum_{k=1}^{\infty}\left(\sum_{|s|=k} e^{-kp_n/n}\right) \le \sum_{r=1}^{\infty}\left(|S|^k e^{-kp_n/n}\right) = \sum_{r=1}^{\infty}\left(|S|e^{-p_n/n}\right)^k < \infty.$$

Let $\pi_n\colon C^*_{\ell^{p_n}}(\Gamma) \to B(\mathcal{H}_n)$ be the GNS representations corresponding to h_n, and let $v_n \in \mathcal{H}_n$ be the canonical cyclic vector. Since since $h_n(s) \to 1$ we see that $\|\pi_n(s)v_n - v_n\| \to 0$, for all $s \in \Gamma$. Hence the direct sum representation $\oplus\pi_n$ weakly contains the trivial representation. It follows that we cannot have $C^*_{\ell^{p_n}}(\Gamma) = C^*_r(\Gamma)$ for all n – otherwise $\oplus\pi_n$ would be defined on $C^*_r(\Gamma)$ and nonamenability prevents the trivial representation from being weakly contained in any representation of $C^*_r(\Gamma)$. $\square$

Remark 4.5. The previous proposition is not optimal; Higson, Ozawa and Okayasu [13] have independently shown that the C^*-algebras $C^*_{\ell^p}(\mathbb{F}_n)$ are mutually non-isomorphic. On the other hand, extracting the crucial ingredients from the proof, we see that the phenomenon presented there is very general. Indeed, suppose that Γ is a non-amenable, a-T-menable group admitting an $\mathbb{N}$-valued conditionally negative type function ψ satisfying an estimate of the following form: there exists $C > 0$ such that for every k we have

$$\#\{\, s \in \Gamma \colon \psi(s) = k \,\} \leq C^k. \tag{2}$$

Taking $h_n(s) = e^{-\psi(s)/n}$ the above proof applies verbatim to show that $C^*_{\ell^p}(\Gamma) \neq C^*_r(\Gamma)$ for some p. This applies, for example, to infinite Coxeter groups – the word length function corresponding to the standard Coxeter generators satisfies the hypothesis for ψ [3].

Remark 4.6. Continuing the previous remark, suppose a non-amenable group Γ acts (cellularly) on a CAT(0) cube complex X. The combinatorial distance d in the one skeleton of X defines an $\mathbb{N}$-valued conditionally negative type function on Γ by

$$\psi(x) = d(x_0, s \cdot x_0),$$

where x_0 is any arbitrarily chosen vertex in X [12].[4] If Γ is finitely generated and the orbit map $s \mapsto s \cdot x_0 : \Gamma \to X$ is a quasi-isometric embedding then the inequality (2) is satisfied. These two conditions hold in many common situations: by the Svarc-Milnor Lemma, they are automatic if the action is proper and cocompact [4] and the complex is finite dimensional; they also hold for the action of Thompson's group F or, more generally, a finitely generated diagram group with *Property B*, on its Farley complex [1].

5. Topological Dynamical Systems

Let Γ be a discrete group, and let X be a compact Hausdorff space on which Γ acts by homeomorphisms. Thinking of transformation groupoids, let $C_c(X \rtimes \Gamma)$ denote the convolution algebra of compactly supported functions on $X \times \Gamma$. We shall represent elements of this algebra as finite formal sums $\sum f_s s$, where each $f_s \in C(X)$; we shall view Γ as a subset of the convolution algebra in the obvious manner.

Definition 5.1. Let $D \triangleleft \ell^\infty(\Gamma)$ be an ideal. A $*$-representation $\pi \colon C_c(X \rtimes \Gamma) \to \mathbb{B}(\mathcal{H})$ is a *D-representation* if $\pi|_\Gamma$ is a D-representation in the sense of Definition 2.1.

Definition 5.2. Let $D \triangleleft \ell^\infty(\Gamma)$ be an ideal. Define a C*-norm $\|\cdot\|_D$ on $C_c(X \rtimes \Gamma)$ by

$$\left\|\sum f_s s\right\|_D := \sup\left\{ \left\|\pi\left(\sum f_s s\right)\right\| : \pi \text{ is a D-representation} \right\},$$

and let $C^*_D(X \rtimes \Gamma)$ denote the completion of $C_c(X \rtimes \Gamma)$ with respect to $\|\cdot\|_D$.

As before, every D-representation extends uniquely to $C^*_D(X \rtimes \Gamma)$; further, $C^*_D(X \rtimes \Gamma)$ admits a *faithful* D-representation. Considering the universal properties, one sees that $C^*_{\ell^\infty}(X \rtimes \Gamma)$ is the universal (or full) crossed product C*-algebra, denoted $C^*(X \rtimes \Gamma)$. It is not clear whether the analogue of Proposition 2.11 holds in the present context.

Recall that a function $h \colon X \times \Gamma \to \mathbb{C}$ is *positive definite* if for each finite set $s_1, \ldots, s_n \in \Gamma$ and point $x \in X$, the matrix

$$[h(s_i.x, s_i s_j^{-1})]_{i,j} \in \mathbb{M}_n(\mathbb{C})$$

[4]While stated only for finite dimensional complexes, the proof given is valid in greater generality.

is positive (semi-definite); here $x \mapsto s.x$ denotes the action of s on X. Recall also that to each positive definite h we can associate a completely positive Schur multiplier

$$\mathrm{m}_h\colon C^*(X \rtimes \Gamma) \to C^*(X \rtimes \Gamma);$$

on finite sums m_h is given by the formula

$$\mathrm{m}_h \left(\sum f_s s \right) = \sum f_s h(s) s$$

where, slightly abusing notation, we have written $h(s) \in C(X)$ for the function $x \mapsto h(x,s)$ [5, Proposition 5.6.16].

Lemma 5.3. *Suppose that $D \lhd \ell^\infty(\Gamma)$ is a translation invariant ideal; suppose that $h\colon X \times \Gamma \to \mathbb{C}$ is positive definite, and that the function $H(s) := \|h(s)\|$ belongs to D. Then, for every state φ on $C^*(X \rtimes \Gamma)$, the GNS representation associated to $\varphi \circ m_h$ is a D-representation.*

Proof. Note that for $f \in \ell^\infty(\Gamma)$ we have, $f \in D \iff |f| \in D$ – this follows from the polar decomposition $f = u|f|$, in which $u \in \ell^\infty(\Gamma)$ is the unitary of 'pointwise rotation'. Also, D is hereditary in the sense that if $0 \le g \le f$ and $f \in D$, then $g \in D$. To see this, define $h \in \ell^\infty(\Gamma)$ to be zero wherever f is, and $h(s) = \frac{g(s)}{f(s)}$ otherwise; evidently $g = hf \in D$.

Now, fix two functions f, $g \in C(X)$ and two group elements s, $t \in \Gamma$. Let v denote the canonical image of fs in the GNS Hilbert space; similarly let w denote the image of gt. Since the linear span of such elements is dense, it suffices to show $\pi_{v,w} \in D$, where π denotes the GNS representation. By the first paragraph, and our assumptions on D and H, it suffices to show $|\pi_{v,w}|$ is bounded above by a constant times some translate of H. This, however, is a straightforward calculation. □

With the previous lemma in hand, the proof of the following result is very similar to its analog in the group case, Theorem 3.2 – use Schur multipliers to approximate arbitrary states by vector states. Whereas Theorem 3.2 was an 'if-and-only-if' statement, we do not know if the converse holds.

Theorem 5.4. *Let $D \lhd \ell^\infty(\Gamma)$ be a translation invariant ideal. Assume there exist positive definite functions $h_n\colon X \times \Gamma \to \mathbb{C}$ satisfying $h_n \to 1$ uniformly on compact sets and for which each $H_n(s) := \|h_n(s)\|$ belongs to D. Then $C^*(X \rtimes \Gamma) = C^*_D(X \rtimes \Gamma)$.* □

Definition 5.5. An action of Γ on X is *amenable* if there exist positive definite functions $h_n \in C_c(X \rtimes \Gamma)$ such that $h_n \to 1$ uniformly on compact sets; it is *a-T-menable* if there exist positive definite functions $h_n \in C_0(X \rtimes \Gamma)$ such that $h_n \to 1$ uniformly on compact sets.

Remark 5.6. Just as every action of an amenable group is amenable, every action of an a-T-menable group is a-T-menable. Indeed, if $h \in c_0(\Gamma)$ is positive definite, then a computation confirms that the function $\tilde{h} \in C_0(X \times \Gamma)$ defined by $\tilde{h}(x,s) = h(s)$ is as well. The assertion now follows easily from the definitions.

Definition 5.7. Let $C^*_{c_c}(X \rtimes \Gamma)$ denote the ideal completion associated to the ideal of finitely supported functions on Γ; let $C^*_{c_0}(X \rtimes \Gamma)$ denote the ideal completion associated to the ideal of functions vanishing at infinity.

We draw several corollaries, the analogs of Corollaries 3.4 and 3.5 in the group case. Again, whereas Corollary 3.4 was an equivalence, the converse of the first corollary is open.

Corollary 5.8. *Let Γ be a discrete group, acting on X. If the action is amenable then $C^*(X \rtimes \Gamma) = C^*_{c_c}(X \rtimes \Gamma)$; if the action is a-T-menable, then $C^*(X \rtimes \Gamma) = C^*_{c_0}(X \rtimes \Gamma)$.* □

Corollary 5.9. *Let Γ be a discrete group, acting on X. If the action is amenable then every covariant representation (that is, $*$-homomorphism $C^*(X \rtimes \Gamma) \to \mathbb{B}(\mathcal{H})$) is weakly contained in a c_c-representation; if the action is a-T-menable then every covariant representation is weakly contained in a c_0-representation.*

Proof. In the case of an amenable action, the hypothesis implies that $C^*(X \rtimes \Gamma) = C^*_{c_c}(X \rtimes \Gamma)$, and $C^*_{c_c}(X \rtimes \Gamma)$ has a faithful c_c-representation. For an a-T-menable action systematically replace $c_c(\Gamma)$ by $c_0(\Gamma)$ throughout. □

Corollary 5.10. *Let Γ be a discrete group acting on X. Let $\pi\colon C^*(X \rtimes \Gamma) \to \mathbb{B}(\mathcal{H})$ be a covariant representation for which $\pi|_\Gamma$ weakly contains the trivial representation of Γ. We have:*

(1) *the action is amenable if and only if Γ is amenable;*
(2) *the action is a-T-menable if and only if Γ is a-T-menable.*

Proof. The 'if' statements are trivial (see Remark 5.6). For the 'only if' statements, let D stand for the appropriate ideal, either $c_c(\Gamma)$ or $c_0(\Gamma)$. Corollary 5.8 implies that $C^*_D(X \rtimes \Gamma) = C^*(X \rtimes \Gamma)$, so that we can form the composition

$$C^*_D(\Gamma) \to C^*_D(X \rtimes \Gamma) = C^*(X \rtimes \Gamma) \to \mathbb{B}(\mathcal{H}).$$

The proof is completed by recalling that Γ is amenable (respectively, a-T-menable) if and only if there is a $c_c(\Gamma)$ (respectively, $c_0(\Gamma)$)-representation of Γ weakly containing the trivial representation. □

To close, we return to the result of Douglas and Nowak described in the introduction. Suppose a discrete group Γ acts on X, and that μ is a quasi-invariant measure on X (in other words, elements of Γ map μ-null sets to μ-null sets). For each element $s \in \Gamma$ the Radon-Nikodym derivative is the non-negative, measurable function ρ_s satisfying

$$\int_X f d\mu = \int_X s.f \rho_s d\mu,$$

for every measurable function f; here, $f \mapsto s.f$ denotes the action of s on f. The ρ_s allow one to construct a covariant representation of $X \rtimes \Gamma$ on the Hilbert space $L^2(X, \mu)$: functions in $C(X)$ act by multiplication and elements $s \in \Gamma$ act by the unitaries

$$U_s(f) := (s.f)\rho^{1/2}.$$

Corollary 5.11. *Let Γ be a discrete group acting on X, with quasi-invariant measure μ. Suppose the representation of Γ on $L^2(X, \mu)$ weakly contains the trivial representation. We have:*

(1) *the action is amenable if and only if Γ is amenable;*
(2) *the action is a-T-menable if and only if Γ is a-T-menable.*

In particular, such an action of a non-a-T-menable group can never be a-T-menable. □

This result, which follows immediately from the previous corollary, generalizes Theorem 3 and Corollary 4 of Douglas and Nowak [8] – the hypotheses of their results imply the existence of a non-zero fixed vector in $L^2(X, \mu)$, namely the square root of $\overline{\rho}$ or $\underline{\rho}$, as appropriate. See [8, Lemma 6].

For invariant probability measures we get an analogue of a well-known amenability result.

Corollary 5.12. *Let Γ be a discrete group acting on X, with invariant probability measure μ. The action is a-T-menable if and only if Γ is a-T-menable. In particular, no measure-preserving action of a non-a-T-menable group can be a-T-menable.*

Proof. The constant functions in $L^2(X, \mu)$ are invariant for Γ. □

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