EXACTNESS AND THE KADISON-KAPLANSKY CONJECTURE

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With affection and admiration, we dedicate this paper to Richard Kadison on the occasion of his ninetieth birthday.

Abstract. We survey results connecting exactness in the sense of C^*-algebra theory, coarse geometry, geometric group theory, and expander graphs. We summarize the construction of the (in)famous non-exact monster groups whose Cayley graphs contain expanders, following Gromov, Arzhantseva, Delzant, Sapir, and Osajda. We explain how failures of exactness for expanders and these monsters lead to counterexamples to Baum-Connes type conjectures: the recent work of Osajda allows us to give a more streamlined approach than currently exists elsewhere in the literature.

We then summarize our work on reformulating the Baum-Connes conjecture using exotic crossed products, and show that many counterexamples to the old conjecture give confirming examples to the reformulated one; our results in this direction are a little stronger than those in our earlier work. Finally, we give an application of the reformulated Baum-Connes conjecture to a version of the Kadison-Kaplansky conjecture on idempotents in group algebras.

1. Introduction

The Baum-Connes conjecture relates, in an important and motivating special case, the topology of a closed, aspherical manifold M to the unitary representations of its fundamental group. Precisely, it asserts that the *Baum-Connes assembly map*

$$K_*(M) \to K_*(C^*_{\mathrm{red}}(\pi_1(M)) \tag{1.1}$$

is an isomorphism from the K-homology of M to the K-theory of the reduced C^*-algebra of its fundamental group. The injectivity and surjectivity of the Baum-Connes assembly map have important implications—injectivity implies that the higher signatures of M are oriented homotopy invariants (the Novikov conjecture), and that M (assumed now to be a

The first author was partially supported by NSF grant DMS-1200475. The second author was partially supported by a grant from the Simons Foundation (#245398). The third author was partially supported by NSF grant DMS-1401126.

spin manifold) does not admit a metric of positive scalar curvature (the Gromov-Lawson-Rosenberg conjecture); surjectivity implies that the reduced C^*-algebra of $\pi_1(M)$ does not contain non-trivial idempotents (the Kadison-Kaplansky conjecture). For details and more information, we refer to [8, Section 7].

Proofs of the Baum-Connes conjecture, or of its variants, generally involve some sort of large-scale or coarse geometric hypothesis on the universal cover of the manifold M. A sample result, important in the context of this piece, is that if the universal cover of M is coarsely embeddable in a Hilbert space, then the Baum-Connes assembly map is injective [77, 64], so that the Novikov conjecture holds for M.

For some time, it was thought possible that every bounded geometry metric space was coarsely embeddable in Hilbert space. At least there was no counterexample to this statement until Gromov made the following assertions [52]: an expander does not coarsely embed in Hilbert space, and there exists a countable discrete group that 'contains' an expander in an appropriate sense. With the appearance of this influential paper, there began a period of rapid progress on counterexamples to the Baum-Connes conjecture [37], and other conjectures. In particular, these so-called *Gromov monster groups* were found to be counterexamples to the Baum-Connes conjecture *with coefficients*; they were also found to be the first examples of *non-exact* groups in the sense of C^*-algebra theory; and expander graphs were found to be counterexamples to the *coarse* Baum-Connes conjecture. We mention that, while counterexamples to most variants of the Baum-Connes conjecture have been found, there is still no known counterexample to the conjecture as we have stated it in (1.1).

The point of view we shall take in this survey is that the failure of exactness, and the failure of the Baum-Connes conjecture (with coefficients) are intimately related. This point of view is not particularly novel—the counterexamples given by Higson, Lafforgue and Skandalis all have the failure of exactness as their root cause [37]. More recent work has moreover suggested that at least some of the counterexamples can be obviated by forcing exactness [57, 74, 75, 21, 25].

We shall exploit this point of view to reformulate the conjecture by replacing the *reduced* C^*-algebra of the fundamental group, and the associated reduced crossed product that is used when coefficients are allowed, on the right hand side by a new group C^*-algebra and crossed product; the new crossed product will by its definition be *exact*. By doing so, we shall undercut the arguments that have lead in the past to the counterexamples, and indeed, we shall see that some of the former counterexamples are *confirming examples* for the new, reformulated conjecture.

To close this introduction, we give a brief outline of the paper. The first several sections are essentially historical. In Sections 2 and 3 we provide background information on exact groups, crossed products, and the group theoretic and coarse geometric properties relevant for the theory surrounding the Baum-Connes and Novikov conjectures. We discuss the relationships among these properties, and their connection to other areas of C^*-algebra theory.

Section 4 contains a detailed discussion of expanders, focusing on the aspects of the theory necessary to produce the counterexamples that will appear in later sections. Here, we follow an approach outlined by Higson in a talk at the 2000 Mount Holyoke conference, but updated to a slightly more modern perspective. For the Baum-Connes conjecture itself, expanders provide counterexamples through the theory of Gromov monster groups. In Section 5 we describe the history and recent progress on the existence of these groups, beginning with the original paper of Gromov and ending with the recent work of Arzhantseva and Osajda.

The remainder of the paper is dedicated to discussion of the implications for the Baum-Connes conjecture itself. We begin in Section 6 by recalling the necessary machinery to define the conjecture, focusing on the aspects needed for the subsequent discussion. In Section 7 we describe how Gromov monster groups give counterexamples to the conjecture, by essentially reducing to the discussion of expanders given in Section 4. In Section 8 we describe how to adjust the right hand side of the Baum-Connes conjecture by replacing the reduced C^*-algebraic crossed product with a new crossed product functor having better functorial properties and in Section 9 we explain how and why this reformulated conjecture outperforms the original by verifying it in the setting of the counterexamples from Section 7. In Section 10 we give an application of the reformulated conjecture to the Kadison-Kaplansky conjecture for the ℓ^1-algebra of a group.

2. Exact groups and crossed products

Throughout, G will be a countable discrete group. Much of what follows makes sense for arbitrary (second countable) locally compact groups, and indeed this is the level of generality we worked at in our original paper [10]. Here, we restrict to the discrete case because it is the most relevant for non-exact groups, and because it simplifies some details.

A *G-C^*-algebra* is a C^*-algebra equipped with an action α of G by $*$-automorphisms. The natural representations for G-C^*-algebras are the *covariant representations*: these consist of a C^*-algebra representation of A as bounded linear operators on a Hilbert space $\mathcal{H}$, together with a unitary representation of G on the same Hilbert space,

$$\pi : A \to \mathcal{B}(\mathcal{H}) \qquad \text{and} \qquad u : G \to \mathcal{B}(\mathcal{H}),$$

satisfying the covariance condition $\pi(\alpha_g(a)) = u_g \pi(a) u_{g^{-1}}$. Essentially, a covariant representation spatially implements the action of G on A.

Crossed products of a G-C^*-algebra A encode both the algebra A and the G-action into a single, larger C^*-algebra. We introduce the notation $A \rtimes_{\mathrm{alg}} G$ for the algebra of finitely supported A-valued functions on G equipped with the following product and involution:

$$f_1 \star f_2(g) = \sum_{h \in G} f_1(h) \alpha_g(f_2(h^{-1}g)) \qquad \text{and} \qquad f^*(t) = \alpha_g(f(g^{-1})^*).$$

The algebra $A \rtimes_{\mathrm{alg}} G$ is functorial for G-equivariant $*$-homomorphisms in the obvious way. We shall refer to $A \rtimes_{\mathrm{alg}} G$ as the *algebraic crossed product* of A by G. Finally, a covariant representation *integrates* to a $*$-representation of $A \rtimes_{\mathrm{alg}} G$ according to the formula

$$\pi \rtimes u(f) = \sum_{g \in G} \pi(f(g)) u_g \ .$$

Two completions of the algebraic crossed product to a C^*-algebra are classically studied: the *maximal* and *reduced* crossed products. The maximal crossed product is the completion of $A \rtimes_{\mathrm{alg}} G$ for the maximal norm, defined by

$$\|f\|_{\max} = \sup\{ \, \|\pi \rtimes u(f)\| : (\pi, u) \text{ a covariant pair} \, \}.$$

Thus, the maximal crossed product has the universal property that every covariant representation integrates (uniquely) to a representation of $A \rtimes_{\max} G$; indeed, it is characterized by this property. The reduced crossed product is defined to be the image of the maximal crossed product in the integrated form of a particular covariant representation. Precisely, fix a faithful and non-degenerate representation π of A on a Hilbert space $\mathcal{H}$ and define a covariant representation

$$\widetilde{\pi} : A \to \mathcal{B}(\mathcal{H} \otimes \ell^2(G)) \qquad \text{and} \qquad \lambda : G \to \mathcal{B}(\mathcal{H} \otimes \ell^2(G))$$

by

$$\widetilde{\pi}(a)(v \otimes \delta_g) = \pi(\alpha_{g^{-1}}(a)) v \otimes \delta_g \qquad \text{and} \qquad \lambda_h(v \otimes \delta_g) = v \otimes \delta_{hg}.$$

The reduced crossed product $A \rtimes_{\mathrm{red}} G$ is the image of $A \rtimes_{\max} G$ under the integrated form of this covariant representation. In other words, $A \rtimes_{\mathrm{red}} G$ is the completion of $A \rtimes_{\mathrm{alg}} G$ for the norm

$$\|f\|_{\mathrm{red}} = \|\widetilde{\pi} \rtimes \lambda(f)\|.$$

The reduced crossed product (and its norm) are independent of the choice of faithful and non-degenerate representation of A. Incidentally, one may check that $\widetilde{\pi} \rtimes \lambda$ is injective on $A \rtimes_{\mathrm{alg}} G$. It follows that the maximal norm is in fact a norm on the algebraic crossed product—no non-zero element has maximal norm equal to zero. In particular, we may view the algebraic crossed product as contained in each of the maximal and reduced crossed products as a dense $*$-subalgebra.

Kirchberg and Wassermann introduced, in their work on continuous fields of C^*-algebras, the notion of an *exact* group [42]. They define a group G to be *exact* if, for every short exact sequence of G-C^*-algebras

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0 \tag{2.1}$$

the corresponding sequence of *reduced* crossed products

$$0 \longrightarrow I \rtimes_{\mathrm{red}} G \longrightarrow A \rtimes_{\mathrm{red}} G \longrightarrow B \rtimes_{\mathrm{red}} G \longrightarrow 0$$

is itself short exact. Several remarks are in order here. First, the map to $B \rtimes_{\mathrm{red}} G$ is always surjective, the map from $I \rtimes_{\mathrm{red}} G$ is always injective, and the composition of the two non-trivial maps is always zero. In other words, exactness of the sequence can only fail in that the image of the map into $A \rtimes_{\mathrm{red}} G$ may be *properly* contained in the kernel of the following map. Second, the sequence obtained by using the *maximal* crossed product (instead of the reduced) is *always* exact; this follows essentially from the universal property of the maximal crossed product.

There is a parallel theory of exact C^*-algebras in which one replaces the reduced *crossed products* by the spatial *tensor products*. In particular, a C^*-algebra D is *exact* if for every short exact sequence of C^*-algebras (2.1)—now without group action—the corresponding sequence

$$0 \longrightarrow I \otimes D \longrightarrow A \otimes D \longrightarrow B \otimes D \longrightarrow 0$$

is exact. Here, we use the spatial tensor product; the analogous sequence defined using the maximal tensor products is always exact, for any D. In the present context, the theories of exact (discrete) groups and exact C^*-algebras are related by the following result of Kirchberg and Wassermann [42, Theorem 5.2].

2.1. **Theorem.** *A discrete group is exact (as a group) precisely when its reduced C^*-algebra is exact (as a C^*-algebra).*

A central notion for us will be that of a *crossed product functor*. By this we shall mean, for each G-C^*-algebra A a completion $A \rtimes_\tau G$ of the algebraic crossed product fitting into a sequence

$$A \rtimes_{\max} G \to A \rtimes_\tau G \to A \rtimes_{\mathrm{red}} G,$$

in which each map is the identity when restricted to $A \rtimes_{\mathrm{alg}} G$ (a dense $*$-subalgebra of each of the three crossed product C^*-algebras). This is equivalent to requiring that the τ-norm dominates the reduced norm on the algebraic crossed product. Further, we require that

$A \rtimes_\tau G$ be functorial, in the sense that if $A \to B$ is a G-equivariant $*$-homomorphism then the associated map on algebraic crossed products extends (uniquely) to a $*$-homomorphism $A \rtimes_\tau G \to B \rtimes_\tau G$.

We shall call a crossed product functor τ *exact* if, for every short exact sequence of G-C^*-algebras (2.1) the associated sequence

$$0 \longrightarrow I \rtimes_\tau G \longrightarrow A \rtimes_\tau G \longrightarrow B \rtimes_\tau G \longrightarrow 0$$

is itself short exact. For example, the maximal crossed product is exact for every group, but the reduced crossed product is exact only for exact groups. We shall see other examples of exact (and non-exact) crossed products later.

3. Some properties of groups, spaces, and actions

In this section, we shall discuss some properties that are important for the study of the Baum-Connes conjecture, and for issues related to exactness: a-T-menability of groups and coarse embeddability of metric spaces, and their relation to various forms of amenability.

The following definition—due to Gromov [26, Section 7.E]—is fundamental for work on the Baum-Connes conjecture.

3.1. **Definition.** A countable discrete group G is *a-T-menable* if it admits an affine isometric action on a Hilbert space $\mathcal{H}$ such that the orbit of every $v \in \mathcal{H}$ tends to infinity; precisely,

$$\|g \cdot v\| \to \infty \Longleftrightarrow g \to \infty$$

Note here that the forward implication is always satisfied; the essential part of the definition is the reverse implication which asserts that as g leaves every finite subset of G the orbit $g \cdot v$ must leave every bounded subset of $\mathcal{H}$.

To discuss the coarse geometric properties relevant for the Novikov conjecture, we must view the countable discrete group G as a metric space. Let us for the moment imagine that G is finitely generated. We fix a finite generating set S, so that every element of G is a finite product, or word, of elements of S and their inverses. We define the associated *word length* by declaring the length of an element g to be the minimal length of such a word; we denote this by $|g|$. This word length function is a *proper length* function, meaning that it is a non-negative real valued function with the following properties:

$$|g| = 0 \text{ iff } g = \text{identity}, \qquad |g^{-1}| = |g|, \qquad |gh| \le |g| + |h|;$$

and infinite subsets of G have unbounded image in $[0, \infty)$. Returning to the general setting, it is well kown (and not difficult to prove) that a countable discrete group admits a proper length function.

We now equip G with a proper length function, for example a word length, and define the associated metric on G by $d(g, h) = |g^{-1}h|$. This metric has *bounded geometry*, meaning that there is a uniform bound on the number of elements in a ball of fixed radius. It is also *left-invariant*, meaning that the the left action of G on itself is by isometries. This metric is not intrinsic to G, but depends on the particular length function chosen. Nevertheless, the identity map is a *coarse equivalence* between any two bounded geometry, left invariant metrics on G. We shall expand on this fact below. Thus, it makes sense to attribute metric properties to G as long as these properties are insentive to coarse equivalence.

Having equipped G with a metric, we are ready to state the following definition, also due to Gromov [26, Section 7.E].[1]

3.2. **Definition.** A countable discrete group G is *coarsely embeddable* (in Hilbert space) if it admits a map $f : G \to \mathcal{H}$ to a Hilbert space such

$$\|f(g) - f(h)\| \to \infty \Longleftrightarrow d(g, h) \to \infty.$$

In this case, f is a *coarse embedding.*

3.3. *Remark.* To relate coarse embeddability and a-T-menability, suppose that G acts on a Hilbert space $\mathcal{H}$ as in Definition 3.1. Fix $v \in \mathcal{H}$ and notice that

$$\|g \cdot v - h \cdot v\| = \|g^{-1}h \cdot v - v\| \sim \|g^{-1}h \cdot v\|$$

(where $\sim$ means differing at most by a universal additive constant). Thus, forgetting the action and recalling that the metric on G has bounded geometry, we see that the orbit map $f(g) = g \cdot v$ is a coarse embedding as in Definition 3.2.

3.4. *Remark.* Only the metric structure of $\mathcal{H}$ enters into the definition of coarse embeddability; the same definition applies equally well to maps from one metric space to another. In this more general setting, a coarse embedding $f : X \to Y$ of metric spaces is a coarse equivalence if for some universal constant C every element of Y is a distance at most C from $f(X)$. It is in this sense that the identity map on G is a coarse equivalence between any two bounded geometry, left invariant metrics. The key point here is that the balls centered at the identity in each metric are finite, so that the length function defining each metric is bounded on the balls for the other.

[1]Gromov originally used the terminology *uniformly embeddable* and *uniform embedding*; this usage has fallen out of favor since it conflicts with terminology from Banach space theory.

To motivate the relevance of these properties for the Baum-Connes and Novikov conjectures suppose, for example, that G acts on a *finite dimensional* Hilbert space as in Definition 3.1. It is then a discrete subgroup of some Euclidean isometry group $\mathbb{R}^n \rtimes O(n)$ (at least up to taking a quotient by a finite subgroup). That such groups satisfy the Baum-Connes conjecture follows already from Kasparov's 1981 conspectus [41, Section 5, Lemma 4], which predates the conjecture itself!

The relevance of the general, infinite dimensional version of a-T-menability, and of coarse embeddability, was apparent to some authors more than twenty years ago. See for example [51, Problems 3 and 4] of Gromov and [68, Problem 3] of Connes. The key technical advance that allowed progress is the *infinite dimensional Bott periodicity* theorem of Higson, Kasparov, and Trout [36]. One has the following theorem: the part dealing with a-T-menability is due to Higson and Kasparov [34, 35], while the part dealing with coarse embeddability is due in main to Yu [77], although with subsequent improvements of Higson [33] and of Skandalis, Tu, and Yu [64].

3.5. **Theorem.** *Let G be a countable discrete group. The Baum-Connes assembly map with coefficients in any G-C^*-algebra A*

$$K_*^{top}(G; A) \to K_*(A \rtimes_{red} G)$$

is an isomorphism if G is a-T-menable, and is split injective if G is coarsely embeddable.

The class of a-T-menable groups is large: it contains all amenable groups as well as free groups, and classical hyperbolic groups; see [22] for a survey. A-T-menability admits many equivalent formulations: the key results are those of Akemann and Walter studying positive definite and negative type functions [1], and of Bekka, Cherix and Valette relating a-T-menabilty as defined above to the properties studied by Akemann and Walter [12] (the latter are usually called the *Haagerup property* due to their appearance in important work of Haagerup [31] on C^*-algebraic approximation results). There are, however, many non a-T-menable groups: the most important examples are those with Kazhdan's *property* (T) such as $SL(3, \mathbb{Z})$: see the monograph [13].

The class of coarsely embeddable groups is huge: as well as all a-T-menable groups, it contains for example all linear groups (over any field) [29] and all Gromov hyperbolic groups [60]. Indeed, for a long time it was unknown whether there existed any group that did not coarsely embed: see for example [26, Page 218, point (b)]. Thanks to expander based techniques which we will explore in later sections, it is now known that non coarsely embeddable groups exist; it is enormously useful here that coarse emeddability makes sense for arbitrary metric spaces, and not just groups.

Before we turn to a discussion of expanders in the next section, we shall describe the close relationship of coarse embeddability to exactness and some other properties of metric spaces, groups and group actions. The key additional idea is that of *Property A*, which was introduced by Yu to be a relatively easily verified criterion for coarse embeddability [77, Section 2]. Property A was quickly realized to be relevant to exactness: see for example [43, Added note, page 556].

All the properties we have discussed so far can be characterized in terms of *positive definite kernels*, and doing so brings the distinctions among them into sharp focus. Recall that a (normalized) positive definite kernel on a set X is a function $f : X \times X \to \mathbb{C}$ satisfying the following properties:

(i) $k(x,x) = 1$ and $k(x,y) = \overline{k(y,x)}$, for all x, $y \in X$;
(ii) for all finite subsets $\{x_1, ..., x_n\}$ of X and $\{a_1, ..., a_n\}$ of $\mathbb{C}$,

$$\sum_{i,j=1}^{n} a_i \overline{a_j} k(x_i, x_j) \geq 0.$$

If we are working with a countable discrete group G we may additionally require the kernel to be left invariant, in the sense that $k(g_1 g, g_1 h) = k(g,h)$ for every g_1, g and $h \in G$.

3.6. **Theorem.** *Let X be a bounded geometry, uniformly discrete metric space. Then X has Property A if and only if for every R (large) and ε (small), there exists a positive definite kernel $k : X \times X \to \mathbb{C}$ such that*

(i) $|1 - k(x,y)| < \epsilon$ whenever $d(x,y) < R$;
(ii) the set $\{\, d(x,y) \in [0,\infty) \colon k(x,y) \neq 0 \,\}$ is bounded;

X is coarsely embeddable if and only if for every R and ε there exists a positive definite kernel k satisfying (i) above, but instead of (ii) the following weaker condition:

(ii)' for every $\delta > 0$ the set $\{\, d(x,y) \in [0,\infty) \colon |k(x,y)| \geq \delta \,\}$ is bounded.

A countable discrete group G is amenable if and only if for every R and ε there exists a left invariant positive definite kernel satisfying (i) and (ii) above; it is a-T-menable if and only if for every R and ε there exists a left invariant positive definite kernel satisfying (i) and (ii)' above.

The characterization of Property A for bounded geometry, uniformly discrete metric spaces in this theorem is due to Tu [66]. It is particularly useful for studying C^*-algebraic approximation properties, in particular exactness, as it can be used to construct so-called *Schur* *multipliers.* The characterization of coarse embeddability can be found in [59, Theorem 11.16]; that of a-T-menability comes from combining [1] and [12]; that of amenability is well known.

The following diagram, in which all the implications are clear from the previous theorem, summarizes the properties we have discussed:

$$\begin{tikzcd} \text{amenability} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Property A} \equiv \text{exactness} \arrow[d, Rightarrow] \\ \text{a-T-menability} \arrow[r, Rightarrow] & \text{coarse embeddability}\ . \end{tikzcd} \tag{3.1}$$

The class of groups with Property A covers all the examples of coarsely embeddable groups mentioned earlier. Indeed, proving the existence of groups without Property A is as difficult as proving the existence groups that do not coarsely embed. Nonetheless, Osajda has recently shown the existence of coarsely embeddable (and even a-T-menable) groups without Property A [56]. In particular, there are no further implications between any of the properties in the diagram.

The following theorem summarizes some of the most important implications relating Property A to C^*-algebra theory.

3.7. **Theorem.** *Let G be a countable discrete group. The following are equivalent:*

(*i*) *G has Property A;*
(*ii*) *G admits an amenable action on a compact space;*
(*iii*) *G is an exact group;*
(*iv*) *the group C^*-algebra $C^*_{red}(G)$ is an exact C^*-algebra.*

The reader can see the survey [72] or [17, Chapter 5] for proofs of most of these results, as well as the definitions that we have not repeated here. The original references are: [38] for the equivalence of (i) and (ii); [30] (partially) and [58] for the equivalence of (i) and (iv); [43, Theorem 5.2] for (iv) implies (iii) as we already discussed in Section 2; and (iii) implies (iv) is easy.

Almost all these implications extend to second countable, locally compact groups with appropriately modified versions of Property A [2, 24, 16]; the exception is (iv) implies (iii), which is an open question in general. Finally, note that Theorem 3.7 has a natural analog in the setting of discrete metric spaces: see Theorem 4.5 below.

3.8. *Remark.* There is an analog of the equivalence of (i) and (ii) for coarsely embeddable groups appearing in [64, Theorem 5.4]: a group is coarsely embeddable if and only if it admits an *a-T-menable action* on a compact space in the sense of Definition 9.2 below.

4. Expanders

In this section, we study *expanders*: highly connected, sparse graphs. Expanders are the easiest examples of metric spaces that do not coarsely embed. They are also connected to K-theory through the construction of *Kazhdan projections*; this construction is at the root of the counterexamples to the Baum-Connes conjecture.

For our purposes, a graph is a *simplicial graph*, meaning that we allow neither loops nor multiple edges. More precisely, a graph Y comprises a (finite) set of vertices, which we also denote Y, and a set of 2-element subsets of the vertex set, which are the edges. Two vertices x and y are *incident* if there is an edge containing them, and we write $x \sim y$ in this case. The number of vertices incident to a given vertex x is its *degree*, denoted $\deg(x)$.

A central object for us is the *Laplacian* of a graph Y, the linear operator $\ell^2(Y) \to \ell^2(Y)$ defined by

$$\Delta f(x) = \deg(x) f(x) - \sum_{y\colon y\sim x} f(y) = \sum_{y\colon y\sim x} f(x) - f(y).$$

A straightforward calculation shows that Δ is a positive operator; in fact

$$\langle \Delta f, f \rangle = \sum_{(x,y)\colon x\sim y} |f(x) - f(y)|^2 \geq 0. \tag{4.1}$$

The kernel of the Laplacian on a connected graph is precisely the space of constant functions. Indeed, it follows directly from the definition that constant functions belong to the kernel; conversely, it follows from (4.1) that if $\Delta f = 0$ and $x \sim y$ then $f(x) = f(y)$, so that f is a constant function (using the connectedness). Thus, the second smallest eigenvalue (including multiplicity) of the Laplacian on a connected graph is strictly positive. We shall denote this eigenvalue by $\lambda_1(Y)$.

An *expander* is a sequence (Y_n) of finite connected graphs with the following properties:

(i) the number of vertices in Y_n tends to infinity, as $n \to \infty$;
(ii) there exists d such that $\deg(x) \leq d$, for every vertex in every Y_n;
(iii) there exists $c > 0$ such that $\lambda_1(Y_n) \geq c$, for every n.

From the discussion above, we see immediately that the property of being an expander is about having both the degree bounded above, and the first eigenvalue bounded away from 0, *independent of* n. The existence of expanders can be proven with routine counting arguments which in fact show that in an appropriate sense *most* graphs are expanders: see [47, Section 1.2]. Nevertheless, the explicit construction of expanders was elusive. The first construction was given by Margulis [49]. Shortly thereafter the close connection with Kazhdan's Property (T) was understood—the collection of finite quotients of a residually finite discrete group with Property (T) are expanders, when equipped with the (Cayley) graph structure coming from a fixed finite generating set of the parent group, and ordered so that their cardinalities tend to infinity [47, Section 3.3]. In particular, the congruence quotients of $\mathrm{SL}(3,\mathbb{Z})$ are an expander.

In the present context, the first counterexamples provided by expanders are to questions in coarse geometry. Given a sequence Y_n of graphs comprising an expander, we consider the associated *box space*, which is a metric space Y with the following properties:

(i) as a set, Y is the disjoint union of the Y_n;
(ii) the restriction of the metric to each Y_n is the graph metric;
(iii) $d(Y_n, Y_m) \to \infty$ for $n \neq m$ and $n+m \to \infty$.

Here, the distance between two vertices in the graph metric is the smallest possible number of edges on a path connecting them. It is not difficult to construct a box space; one simply declares that the distance from a vertex in Y_m to a vertex in the union of the Y_n for $n < m$ is sufficiently large. Further, the identity map provides a coarse equivalence (see Remark 3.4) between any two box spaces, so that their coarse geometry is well defined.

4.1. **Proposition.** *A box space associated to an expander sequence is not coarsely embeddable, and hence does not have property A.*

This proposition was originally stated by Gromov, and proofs were later supplied by several authors including Higson and Dranishnikov: see for example [59, Proposition 11.29]. More recently, many results of this type have been proven, primarily, negative results about the impossibility of coarsely embedding various types of expanders in various types of Banach space, and other non-positively curved spaces. See for example [46, 50, 44].

On the analytic side, expanders have also proven useful for counterexamples, essentially because of the presence of *Kazhdan type projections*. On a single, connected, finite graph Y, we have the projection p onto the constant functions, which is to say, onto the kernel of the Laplacian. This projection can be obtained as a spectral function of the Laplacian; precisely, $p = f(\Delta)$ provided that $f(0) = 1$ and that $f \equiv 0$ on the remaining eigenvalues of Δ.

Now suppose that Y is the box space of a sequence Y_n of finite graphs with uniformly bounded vertex degrees. We can then consider the operators

$$p = \begin{pmatrix} p_1 & 0 & 0 & \dots \\ 0 & p_2 & 0 & \dots \\ 0 & 0 & p_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} \Delta_1 & 0 & 0 & \dots \\ 0 & \Delta_2 & 0 & \dots \\ 0 & 0 & \Delta_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.2}$$

acting on $\ell^2(Y)$, identified with the direct sum of the spaces $\ell^2(Y_n)$. While p will not generally be a spectral function of Δ, it will be when Y_n is an expander sequence. Indeed, in this case if f is a continuous function on $[0, \infty)$ satisfying $f(0) = 1$ and $f \equiv 0$ on $[c, \infty)$ then $p = f(\Delta)$. We shall refer to p as the *Kazhdan projection* of the expander.

The importance of the Kazhdan projection is difficult to overstate: one can often show that Kazhdan projections are not in the range of Baum-Connes type assembly maps, and are therefore fundamental for counterexamples. This is best understood in the context of metric spaces, and to proceed we need to introduce the coarse geometric analog of the group C^*-algebra. For convenience, we consider here only the *uniform Roe algebra* of a (discrete) metric space X. A bounded operator T on $\ell^2(X)$ has *finite propagation* if there exists $R > 0$ such that T cannot propagate signals over a distance greater than R: precisely, for every finitely supported function f on X, the support of $T(f)$ is contained in the R-neighborhood of the support of f. The collection of all bounded operators having finite propagation is a $*$-algebra, and its closure is the uniform Roe algebra of X. We denote the uniform Roe algebra of X by $C^*(X)$, and remark that it contains the compact operators on $\ell^2(X)$ as an ideal.

4.2. **Proposition.** *Let Y be the box space of an expander sequence. The Kazhdan projection p is not compact, and belongs to $C^*(Y)$.*

Proof. The Kazhdan projection has infinite rank; it projects onto the space of functions that are constant on each Y_n. Further, the Laplacian propagates signals a distance at most 1, so that both Δ and its spectral function $p = f(\Delta)$ belong to the C^*-algebra $C^*(Y)$. $\square$

The Kazhdan projection of an expander Y has another significant property: it is a *ghost*. Here, returning to a discrete metric space X, a *ghost* is an element $T \in C^*(X)$ whose 'matrix entries tend to 0 at infinity'; precisely, the suprema

$$\sup_{z \in X} |T_{xz}| \quad \text{and} \quad \sup_{z \in X} |T_{zx}|$$

of matrix entries over the 'x^{th} row' and 'x^{th} column' tend to zero as x tends to infinity. With this definition it is immediate that compact operators are ghosts, and easy to see that a finite propagation operator is a ghost precisely when it is compact. The Kazhdan projection in $C^*(Y)$ is a (non-compact!) ghost because the elements p_n in its matrix representation (4.2) are

$$p_n = \frac{1}{\mathrm{card}(Y_n)} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

To understand the importance of ghostliness we recast the definition slightly. We shall denote the Stone-Cech compactification of X by βX, and shall identify its elements with *ultrafilters* on X. Each element of X gives rise to an ultrafilter, so that $X \subset \beta X$. We shall be primarily concerned with the *free* ultrafilters, that is, the elements of the *Stone-Cech corona* $\beta_\infty X = \beta X \setminus X$. A bounded function ϕ on X has a limit against each ultrafilter. If an ultrafilter corresponds to a point of X this limit is simply the evaluation of ϕ at that point; if ω is a free ultrafilter, we shall denote the limit by $\omega\text{-lim}(\phi)$.

Suppose now we are given a (free) ultrafilter $\omega \in \beta_\infty X$. We define a linear functional on $C^*(X)$ by the formula

$$\Omega(T) = \omega\text{-lim}(x \mapsto T_{xx}).$$

Here, the T_{xx} are the diagonal entries of the matrix representing the operator T in the standard basis of $\ell^2(X)$. We check that Ω is a state on $C^*(X)$. Indeed, one checks immediately that $\Omega(1) = 1$, and a simple calculation shows that the diagonal entries of T^*T are given by

$$(T^*T)_{xx} = \sum_z |T_{zx}|^2\,; \tag{4.3}$$

thus they are non-negative so their limit is as well. Finally, we define $C^*_\infty(X)$ to be the image of $C^*(X)$ in the direct sum of the Gelfand-Naimark-Segal representations of the states defined in this way from free ultrafilters ω; this is a quotient of $C^*(X)$. While the following proposition is well known, not being able to locate a proof in the literature we provide one here.

4.3. **Proposition.** *The kernel of the $*$-homomorphism $C^*(X) \to C^*_\infty(X)$ is the set of all ghosts.*

Proof. Suppose T is a ghost. Since the ghosts form an ideal in $C^*(X)$ we have that for every R and $S \in C^*(X)$ the product RTS is also a ghost. In particular, its on-diagonal matrix entries $(RTS)_{xx}$ tend to zero as $x \to \infty$, so that their limit against every free ultrafilter is also zero. This mean that the norm of T in the GNS representation associated to every free ultrafilter is zero, so that T maps to zero in $C^*_\infty(X)$.

Conversely, suppose that T maps to zero in $C^*_\infty(X)$, so that T^*T does as well. Hence the limit of the on-diagonal matrix entries $(T^*T)_{xx}$ is zero against every free ultrafilter, so that they converge to zero in the ordinary sense as $x \to \infty$. Now, according to (4.3) we have

$$(T^*T)_{xx} = \sum_z |T_{zx}|^2 \geq \sup_z |T_{zx}|^2,$$

so that $\sup_z |T_{zx}|^2 \to 0$ as $x \to \infty$ as well. Applying the same argument to TT^* shows that $\sup_z |T_{xz}|^2$ tends to 0 as $x \to \infty$, and thus T is a ghost. □

Putting everything together, we have for the box space Y of an expander a short sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(Y) \longrightarrow C^*_\infty(Y) \longrightarrow 0. \tag{4.4}$$

The sequence is *not* exact because the Kazhdan projection belongs to the kernel of the quotient map, although it is not compact. As we shall now show, it is possible to detect the K-theory class of the Kazhdan projection and to see that the sequence is not exact even at the level of K-theory. We shall see in Section 7 that this is the phenomenon underlying the counterexamples to the Baum-Connes conjecture.

4.4. **Proposition.** *The K-theory class of the Kazhdan projection is not in the image of the map $K_0(\mathcal{K}) \to K_0(C^*(Y))$.*

Proof. We have, for each 'block' Y_n a contractive linear map $C^*(Y) \to \mathcal{B}(\ell^2(Y_n))$ defined by cutting down by the appropriate projection. These are asymptotically multiplicative on the algebra of finite propagation operators, and we obtain a $*$-homomorphism

$$C^*(Y) \to \frac{\prod_n \mathcal{B}(\ell^2(Y_n))}{\oplus_n \mathcal{B}(\ell^2(Y_n))}. \tag{4.5}$$

Taking the rank in each block (equivalently, taking the map on K-theory induced by the canonical matrix trace on $\mathcal{B}(\ell^2(Y_n))$) gives a homomorphism

$$K_0\left(\prod_n \mathcal{B}(\ell^2(Y_n))\right) \to \prod_n \mathbb{Z}. \tag{4.6}$$

As $K_1(\oplus_n \mathcal{B}(\ell^2(Y_n)) = 0$, the six term exact series in K-theory specializes to an exact sequence

$$K_0(\oplus_n \mathcal{B}(\ell^2(Y_n))) \longrightarrow K_0(\prod_n \mathcal{B}(\ell^2(Y_n))) \longrightarrow K_0\left(\frac{\prod_n \mathcal{B}(\ell^2(Y_n))}{\oplus_n \mathcal{B}(\ell^2(Y_n))}\right) \longrightarrow 0\ .$$

Composing the 'rank homomorphism' in line (4.6) with the quotient map $\prod_n \mathbb{Z} \to \prod_n \mathbb{Z}/\oplus_n\mathbb{Z}$ clearly annihilates the image of $K_0(\oplus_n\mathcal{B}(\ell^2(Y_n)))$, and thus from the sequence above gives rise to a homomorphism

$$K_0\left(\frac{\prod_n \mathcal{B}(\ell^2(Y_n))}{\oplus_n\mathcal{B}(\ell^2(Y_n))}\right) \to \frac{\prod_n \mathbb{Z}}{\oplus_n\mathbb{Z}}.$$

Finally, combining with the K-theory map induced by the $*$-homomorphism in line (4.5) gives a group homomorphism

$$K_0(C^*(Y)) \to \frac{\prod_n \mathbb{Z}}{\oplus_n\mathbb{Z}}.$$

Any K-theory class in the image of the map $K_0(\mathcal{K}) \to K_0(C^*(Y))$ goes to zero under the map in the line above. On the other hand, the Kazhdan projection restricts to a rank one projection on each $\ell^2(Y_n)$ and therefore its image is $[1, 1, 1, 1, ...]$, and so is non-zero. $\square$

The failure of the above sequence (4.4) to be exact is at the base of many failures of exactness and other approximation properties in operator theory and operator algebras. See for example the results of Voiculescu [70] and Wassermann [71]. More recently, the results in the following theorem (an analog of Theorem 3.7 for metric spaces) have been filled in, clarifying the relationship between Property A, ghosts, amenability and exactness. Note that the box space of an expander, or a countable discrete group with proper, left invariant metric, satisfy the hypotheses.

4.5. **Theorem.** *Let X be a bounded geometry uniformly discrete metric space. Then the following are equivalent:*

(*i*) *X has Property A;*
(*ii*) *the coarse groupoid associated to X is amenable;*
(*iii*) *the uniform Roe algebra $C^*(X)$ is an exact C^*-algebra;*
(*iv*) *all ghost operators are compact.*

These results can be found in the following references: [64, Theorem 5.3] for the equivalence of (i) and (ii), and the definition of the coarse groupoid; [62] for the equivalence of (i) and (iii); [59, Proposition 11.4.3] for (i) implies (iv); and [61] for (iv) implies (i).

5. Gromov monster groups

As mentioned in the introduction, the search for counterexamples to the Baum-Connes conjecture began in earnest with the provocative remarks found in the last section of [52]. There, Gromov describes a model for a *random presentation of a group* and asserts that under certain conditions such a random group will almost surely *not* be coarsely embeddable in

Hilbert space, or in any ℓ^p-space for finite p. The non-embeddable groups arise by randomly labeling the edges of a suitable expander family with labels that correspond to the generators of a given, for example, free group. For the method to work, it is necessary that the expander have *large girth*: the length of the shortest cycle in the n^{th} constituent graph tends to infinity with n. Thus labeled, cycles in the expander graphs give words in the generators which are viewed as relators in an (infinite) presentation of a random group. Gromov then goes on to state that a further refinement of the method would reveal that certain of these random and non-embeddable groups are themselves subgroups of finitely presented groups, which are therefore also non-embeddable. More details appeared in the subsequent paper of Gromov [27], and in the further work of Arzhantseva and Delzant [3].

From the above sketch given by Gromov, it is immediately clear that the original expander graphs Y_n would in some sense be 'contained' in the Cayley graph of the random group G. And groups 'containing expanders' became known as *Gromov monsters*. As is clearly explained in a recent paper of Osajda [56], it is an inherent limitation of Gromov's method that the expanders will *not* themselves be coarsely embedded in the random group. Rather, they will be 'contained' in the following weaker sense: there exist constants a, b and c_n such that c_n is much smaller than the diameter of Y_n, and such that for each n there exists a map $f_n : Y_n \to G$ satisfying

$$bd(x, y) - c_n \leq d(f(x), f(y)) \leq ad(x, y).$$

In other words, each Y_n is quasi-isometrically embedded in the Cayley graph of G, but the additive constant involved in the lower bound decays as $n \to \infty$. This is nevertheless sufficient for the non-embeddability of G, and for the counterexamples of Higson, Lafforgue and Skandalis [37] (who in fact use a still weaker form of 'containment').

In part as a matter of convenience, and in part out of necessity, we shall adopt the following more restricted notion of Gromov monster group.

5.1. **Definition.** A *Gromov monster* (or simply *monster*) group is a discrete group G, equipped with a fixed finite generating set and which has the following property: there exists a subset Y of G which is isometric to a box space of a large girth, constant degree, expander.

Here, it is equivalent to require that each of the individual graphs Y_n comprising the expander are isometrically embedded in G; using the isometric action of G on itself, it is straightforward to arrange the Y_n (rather, their images in G) into a box space.

Building on earlier work with Arzhantseva [5], groups as in this definition were shown to exist by Osajda: see [56, Theorem 3.2]. We recall in rough outline the method. The basic data is a sequence of finite, connected graphs Y_n of uniformly finite degree satisfying the following conditions:

(i) $\mathrm{diam}(Y_n) \to \infty$;
(ii) $\mathrm{diam}(Y_n) \leq A\,\mathrm{girth}(Y_n)$, for some constant A independent of n;
(iii) $\mathrm{girth}(Y_n) \leq \mathrm{girth}(Y_{n+1})$, and $\mathrm{girth}(Y_1) > 24$.

Here, recall that the *girth* of a graph is the length of the shortest simple cycle. While the method is more general, in order to construct monster groups the Y_n will, of course, be taken to be a suitable family of expanders. These conditions are less restrictive than those originally proposed by Gromov, and in his paper Osajda describes an explicit set of expanders that satisfy them.

Using a combination of combinatorial and probabilistic arguments, Osajda produces two labelings of the edges in the individual Y_n with letters from a finite alphabet: one satisfies a small cancellation condition for pieces from different blocks and the other for pieces from a common block. He then combines these in a straightforward way to obtain a labeling that globally satisfies the $C'(1/24)$ small cancellation condition. The monster group G is the quotient of the free group on the letters used in the final labeling by the normal subgroup generated by the relations read along the cycles of the graphs Y_n. It was known from previous work that the $C'(1/24)$ condition implies that the individual Y_n will be isometrically embedded in the Cayley graph of G [5, 55].

The infinitely presented Gromov monsters described here may seem artificial. After all, in the introduction we formulated the Baum-Connes conjecture for fundamental groups of closed aspherical manifolds, and one may prefer to confine attention to finitely presented groups. Fully realizing Gromov's original statement, Osajda remarks that a general method developed earlier by Sapir [63] leads to the existence of closed, aspherical manifolds whose universal covers exhibit similar pathologies. Summarizing, we have the following result.

5.2. **Theorem.** *Gromov monster groups* (*in the sense of Definition 5.1*) *exist. Further, there exist closed aspherical manifolds whose fundamental groups contain quasi-isometrically embedded expanders.*

While groups as in the second statement of this theorem would not qualify as Gromov monster groups under our restricted definition above, their existence is very satisfying.

We shall close this section with a more detailed discussion of the relationship between the properties introduced in Section 3. As we mentioned previously, none of the implications in diagram (3.1) is reversible. The most difficult point concerns the existence of discrete groups (or even bounded geometry metric spaces) that are coarsely embeddable but do not have Property A. The first example of such a *space* was given by Arzhantseva, Guentner and Spakula [4] (non-bounded geometry examples were given earlier by Nowak [54]); their space is the box space in which the blocks are the iterated $\mathbb{Z}/2$-homology covers of the figure-8 space, i.e. the wedge of two circles.

In the case of groups, a much more ambitious problem is the existence of a discrete group which is a-T-menable, but does not have Property A. Building on earlier work with Arzhantseva, this problem was recently solved by Osajda [5, 56]. The strategy is similar to the construction of Gromov monsters: use a graphical small cancellation technique to embed large girth graphs with uniformly bounded degree (at least 3) into the Cayley graph of a finitely generated group. Again, under the $C'(1/24)$ hypothesis the graphs will be isometrically embedded. The large girth hypothesis and the assumption that each vertex has degree at least 3 ensure, by a result of Willett [73], that the group will not have Property A.

The remaining difficulty is to show that the group constructed is a-T-menable under appropriate hypotheses on the graphs and the labelling. The key idea is due to Wise, who showed that certain finitely presented classical small cancellation groups are a-T-menable, by endowing their Cayley graphs with the structure of a *space with walls* for which the wall pseudometric is proper [76].

6. The Baum-Connes conjecture with coefficients

When discussing the Baum-Connes conjecture in the introduction, we considered it as a *higher index map*

$$K_*(M) \to K_*(C^*_{\mathrm{red}}(G)), \tag{6.1}$$

which takes the K-homology class defined by an elliptic differential operator D on a closed aspherical manifold M with fundamental group G to the higher index of D in the K-theory of the reduced group C^*-algebra of G. This point of view is perhaps the most intuitive way to view the conjecture and also leads to some of its most important applications. Here however, we need to get 'under the hood' of the Baum-Connes machinery, and give enough definitions so that we can explain our constructions.

To formulate the conjecture more generally, and in particular to allow coefficients in a G-C^*-algebra, it is usual to use bivariant K-theory and the notion of *descent*. Even if one is only interested in the classical conjecture of (6.1), the extra generality is useful as it grants access to many powerful tools, and has much better naturality and permanence properties under standard operations on groups. There are two standard bivariant K-theories available:

the KK-theory of Kasparov, and the E-theory of Connes and Higson. These two theories have similar formal properties, and for our purposes, it would not make much difference which theory we use (see Remark 8.2 below). However, at the time we wrote our paper [10] it was only clear how to make our constructions work in E-theory, and for the sake of consistency we use E-theory here as well.

We continue to work with a countable discrete group G. We shall denote the category whose objects are G-C^*-algebras and whose morphisms are equivariant $*$-homomorphisms by $\mathbf{GC}^*$; similarly, $\mathbf{C}^*$ denotes the category whose objects are C^*-algebras and whose morphisms are $*$-homomorphisms. Further, we shall assume that all C^*-algebras are separable.

The *equivariant E-theory category*, defined in [28] and which we shall denote $\mathbf{E^G}$, is obtained from the category $\mathbf{GC}^*$ by appropriately enlarging the morphism sets. More precisely, the objects of $\mathbf{E^G}$ are the G-C^*-algebras. An equivariant $*$-homomorphism $A \to B$ gives a morphism in $\mathbf{E^G}$ and further, there is a covariant functor from $\mathbf{GC}^*$ to $\mathbf{E^G}$ that is the identity on objects. We shall denote the morphisms sets in $\mathbf{E^G}$ by $E^G(A, B)$. These are abelian groups, and it follows that for a fixed G-C^*-algebra B, the assignments

$$A \mapsto E^G(A, B) \qquad \text{and} \qquad A \mapsto E^G(B, A)$$

are, respectively, a contravariant and a covariant functor from $\mathbf{GC}^*$ to the category of abelian groups.

Let now $\underline{E}G$ denote a universal space for proper actions of G; this means that $\underline{E}G$ is a metrizable space equipped with a proper G-action such that the quotient space is also metrizable, and moreover that any metrizable proper G-space admits a continuous equivariant map into $\underline{E}G$, which is unique up to equivariant homotopy. Such spaces always exist [8]. Suppose $X \subseteq \underline{E}G$ is a G-invariant and cocompact subset; this means that X is closed and that there is a compact subset $K \subseteq \underline{E}G$ such that $X \subseteq G \cdot K$. Such an X is locally compact (and Hausdorff), and if $X \subseteq Y$ are two such subsets of $\underline{E}G$ there is an equivariant $*$-homomorphism $C_0(Y) \to C_0(X)$ defined by restriction. In this way the various $C_0(X)$, with X ranging over the G-invariant and cocompact subsets of $\underline{E}G$, becomes a directed set of G-C^*-algebras and equivariant $*$-homomorphisms.

It follows from this discussion that for any G-C^*-algebra A we may form the direct limit

$$K_0^{\mathrm{top}}(G; A) := \lim_{\substack{X \subseteq \underline{E}G \\ X \text{ cocompact}}} E^G(C_0(X); A),$$

and similarly for K_1 using suspensions. The universal property of $\underline{E}G$ together with homotopy invariance of the E-theory groups implies that $K_*^{\mathrm{top}}(G; A)$ does not depend on the choice of $\underline{E}G$ up to unique isomorphism. It is called the *topological K-theory of G*. This group will be the domain of the Baum-Connes assembly map.

To define the assembly map, we need to discuss descent. Specializing the construction of the equivariant E-theory category to the trivial group gives the *E-theory category*, which we shall denote by $\mathbf{E}$. The objects in this category are the C^*-algebras, and the morphisms from A to B are an abelian group denoted $E(A, B)$. A $*$-homomorphism $A \to B$ gives a morphism in this category, and there is a covariant functor from the category of C^*-algebras and $*$-homomorphisms to $\mathbf{E}$ that is the identity on objects. Moreover for any C^*-algebra B, the group $E(\mathbb{C}, B)$ identifies naturally with the K-theory group $K_0(B)$.

Recall from Section 2 that the maximal crossed product defines a functor from the category $\mathbf{GC}^*$ to the category $\mathbf{C}^*$. The following theorem asserts that it is possible to extend this functor to the category $\mathbf{E^G}$, so that it becomes defined on the generalized morphisms belonging to $\mathbf{E^G}$ but not to $\mathbf{GC}^*$: see [28, Theorem 6.22] for a proof.

6.1. **Theorem.** *There is a (maximal) descent functor $\rtimes_{max} : \mathbf{E^G} \to \mathbf{E}$ which agrees with the usual maximal crossed product functor both on objects and on morphisms in $\mathbf{E^G}$ coming from equivariant $*$-homomorphisms.*

To complete the definition of the Baum-Connes assembly map, we need to know that if X is a locally compact, proper and cocompact G-space, then $C_0(X) \rtimes_{\max} G$ contains a *basic projection*, denoted p_X, with properties as in the next result: see [28, Chapter 10] for more details.

6.2. **Proposition.** *Let X be a locally compact, proper, cocompact G-space. The K-theory class of the basic projection $[p_X] \in K_0(C_0(X) \rtimes_{max} G) = E(\mathbb{C}, C_0(X) \rtimes_{max} G)$ has the following properties:*

(i) *$[p_X]$ depends only on X (and not on choices made in the definition of p_X);*
(ii) *$[p_X]$ is functorial for equivariant maps.*

Here, functoriality means that if $X \to Y$ is an equivariant map of spaces as in the statement of the proposition, then the classes $[p_X]$ and $[p_Y]$ correspond under the functorially induced map on K-theory.

Now, let X be a proper, locally compact G-space and let A be a G-C^*-algebra. The *assembly map for X with coefficients in A* is defined as the composition

$$\begin{aligned} E^G(C_0(X), A) &\to E(C_0(X) \rtimes_{\max} G, A \rtimes_{\max} G) \\ &\to E(\mathbb{C}, A \rtimes_{\max} G) \\ &\to E(\mathbb{C}, A \rtimes_{\mathrm{red}} G), \end{aligned}$$

in which the first arrow is the descent functor, the second is composition in $\mathbf{E}$ with the basic projection, and the third is induced by the quotient map $A \rtimes_{\max} G \to A \rtimes_{\mathrm{red}} G$. It follows now from property (ii) in Proposition 6.2 that if $X \to Y$ is an equivariant inclusion of locally compact, proper, cocompact G-spaces, then the diagram

$$\begin{tikzcd} E(C_0(X) \rtimes_{\max} G, A \rtimes_{\max} G) \arrow[r] \arrow[d] & E(\mathbb{C}, A \rtimes_{\max} G) \arrow[d, equal] \\ E(C_0(Y) \rtimes_{\max} G, A \rtimes_{\max} G) \arrow[r] & E(\mathbb{C}, A \rtimes_{\max} G) \end{tikzcd}$$

commutes. Here, the horizontal arrows are given by composition with the appropriate basic projections, and the left hand vertical arrow is composition with the $*$-homomorphism $C_0(Y) \rtimes_{\max} G \to C_0(X) \rtimes_{\max} G$ induced by the inclusion $X \to Y$. Hence the assembly maps are compatible with the direct limit defining $K_0^{\mathrm{top}}(G; A)$, and give a well-defined homomorphism

$$K_0^{\mathrm{top}}(G; A) \to E(\mathbb{C}, A \rtimes_{\mathrm{red}} G) = K_0(A \rtimes_{\mathrm{red}} G).$$

Everything works similarly on the level of K_1 using suspensions, and thus we get a homomorphism

$$\mu : K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_{\mathrm{red}} G),$$

which is, by definition, the *Baum-Connes assembly map.* The *Baum-Connes conjecture* states that this map is an isomorphism.

6.3. *Remark.* Following through the construction above without passing through the quotient to the reduced crossed product gives the *maximal Baum-Connes assembly map*

$$\mu : K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_{\max} G).$$

It plays an important role in the theory, but is known *not* to be an isomorphism in general thanks to obstructions that exist whenever G has Kazhdan's *property* (T) [13]; we will come back to this point later.

7. Counterexamples to the Baum-Connes conjecture

In this section, we discuss a class of counterexamples to the Baum-Connes conjecture with coefficients. These are based on [37, Section 7] and [74, Section 8], but are a little simpler and more concrete than others appearing in the literature. The possibility of a simpler construction comes down to the straightforward way the monster groups constructed by Osajda contain expanders.

The existence of counterexamples depends on the following key fact: the left and right hand sides of the Baum-Connes conjecture see short exact sequences of G-C^*-algebras differently. To see this, note that the properties of E-theory as discussed in Section 6 imply that the Baum-Connes assembly map is functorial in the coefficient algebra: precisely, an equivariant $*$-homomorphism $A \to B$ induces a commutative diagram

$$\begin{tikzcd} K_*^{\mathrm{top}}(G;A) \arrow[r] \arrow[d] & K_*(A \rtimes_{\mathrm{red}} G) \arrow[d] \\ K_*^{\mathrm{top}}(G;B) \arrow[r] & K_*(B \rtimes_{\mathrm{red}} G), \end{tikzcd}$$

in which the horizontal maps are the Baum-Connes assembly maps, and the vertical maps are induced from the associated morphism $A \to B$ in the equivariant E-theory category. The following lemma gives a little more information when the maps come from a short exact sequence.

7.1. **Lemma.** *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of separable G-C^*-algebras. There is a commutative diagram of Baum-Connes assembly maps

$$\begin{tikzcd} K_0^{top}(G;I) \arrow[r] \arrow[d] & K_0^{top}(G;A) \arrow[r] \arrow[d] & K_0^{top}(G;B) \arrow[d] \\ K_0(I \rtimes_{red} G) \arrow[r] & K_0(A \rtimes_{red} G) \arrow[r] & K_0(B \rtimes_{red} G), \end{tikzcd}$$

in which the horizontal arrows are the functorially induced ones. Moreover, the top row is exact in the middle.

Proof. The existence and commutativity of the diagram follows from our discussion of E-theory. Exactness of the top row follows from exactness properties of E-theory (see [28, Theorem 6.20]) and the fact that exactness is preserved under direct limits. $\square$

The following consequence of the Baum-Connes conjecture with coefficients is immediate from the lemma.

7.2. **Corollary.** *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of separable G-C^*-algebras. If the Baum-Connes conjecture for G with coefficients in all of I, A, and B is true then the corresponding sequence of K-groups

$$K_0(I \rtimes_{red} G) \longrightarrow K_0(A \rtimes_{red} G) \longrightarrow K_0(B \rtimes_{red} G)$$

is exact in the middle.

We will now use Gromov monster groups to give a concrete family of examples where this fails, thus contradicting the Baum-Connes conjecture with coefficients. Assume that G is a monster as in Definition 5.1. In particular, there is assumed to be a subset $Y \subseteq G$ which is (isometric to) a large girth, constant degree expander. The essential idea is to relate Proposition 4.4 to appropriate crossed products.

To do this, equip $\ell^\infty(G)$ with the action induced by the *right* translation action of G on itself. Consider the (non-unital) G-invariant C^*-subalgebra of $\ell^\infty(G)$ generated by the functions supported in Y; as this C^*-algebra is commutative, we may as well write it as $C_0(W)$ where W is its spectrum, a locally compact G-space. Note that as G acts on itself by isometries on the *left*, the *right* action of $g \in G$ moves elements by distance exactly the length $|g|$: in symbols, $d(x, xg) = |g|$ for all $x \in G$. Hence $C_0(W)$ is the closure of the $*$-subalgebra of $\ell^\infty(G)$ consisting of all functions supported within a finite distance from Y. It follows that $C_0(W)$ contains $C_0(G)$ as an essential ideal, whence G is an open dense subset of W. Defining $\partial W := W \setminus G$, it follows that $C_0(\partial W) = C_0(W)/C_0(G)$.

Let ρ denote the right regular representation of G on $\ell^2(G)$ and M the multiplication action of $\ell^\infty(G)$ on $\ell^2(G)$. Then the pair (M, ρ) is a covariant representation of $\ell^\infty(G)$ for the right G-action. Moreover, it is well-known (compare for example [17, Proposition 5.1.3]) that this pair integrates to a faithful representation of $\ell^\infty(G) \rtimes_{\mathrm{red}} G$ on $\ell^2(G)$ that takes $C_0(G) \rtimes_{\mathrm{red}} G$ onto the compact operators. As the reduced crossed product preserves inclusions, it makes sense to restrict this representation to $C_0(W) \rtimes_{\mathrm{red}} G$, thus giving a faithful representation of $C_0(W) \rtimes_{\mathrm{red}} G$ on $\ell^2(G)$.

The key facts we need to build our counterexamples are contained in the following lemma. To state it, let $C^*(Y)$ denote the uniform Roe algebra of Y and $C^*_\infty(Y)$ the quotient as in Section 4. Represent $C^*(Y)$ on $\ell^2(G)$ by extending by zero on the orthogonal complement $\ell^2(G \setminus Y)$ of $\ell^2(Y)$.

7.3. **Lemma.** *The faithful representations of $C_0(W)\rtimes_{red}G$ and $C^*(Y)$ on $\ell^2(G)$ defined above give rise to a commutative diagram*

$$\begin{tikzcd} \mathcal{K}(\ell^2(Y)) \arrow[r] \arrow[d] & C^*(Y) \arrow[r] \arrow[d] & C^*_\infty(Y) \arrow[d] \\ C_0(G) \rtimes_{red} G \arrow[r] & C_0(W) \rtimes_{red} G \arrow[r] & C_0(\partial W) \rtimes_{red} G \end{tikzcd}$$

where the vertical arrows are all inclusions of subsets of the bounded operators on $\ell^2(G)$. Moreover, the vertical arrows are all inclusions of full corners.

Proof. Let χ denote the characteristic function of Y, considered as an element of $C_0(W)$. Our first goal is to identify the C^*-algebras in the top row of the diagram with the corners of those in the botom row corresponding to the projection χ. We begin with the C^*-algebra $C_0(W)\rtimes_{\mathrm{red}}G$, which is generated by operators of the form $f\rho_g$, where $f\in C_0(W)$ and $g\in G$. The compression of such an operator

$$\chi(f\rho_g)\chi : \ell^2(Y)\to\ell^2(Y) \tag{7.1}$$

has matrix coefficients

$$\langle\, \delta_x, \chi f\rho_g\chi(\delta_y)\,\rangle = \langle\, \delta_x, f\rho_g(\delta_y)\,\rangle = (f\delta_{yg^{-1}})(x) = \begin{cases} f(x), & y = xg \\ 0, & \text{else,} \end{cases} \tag{7.2}$$

for x, $y\in Y$. As discussed above $d(x,xg)=|g|$, so that the operator in line (7.1) has finite propagation (at most $|g|$). Hence the corner $\chi(C_0(W)\rtimes_{\mathrm{red}}G)\chi$ is contained in $C^*(Y)$.

Conversely, suppose T is a finite propagation operator on $\ell^2(Y)$. For each $g\in G$ define a complex valued function f_g on G by

$$f_g(x) = \begin{cases} \langle\, \delta_x, T\delta_{xg}\,\rangle, & x,\ xg\in Y \\ 0, & \text{else.} \end{cases}$$

Now, f_g is identically $=0$ if $|g|$ is greater than the propagation of T, and an elementary check of matrix coefficients using line (7.2) shows that T is given by the (finite) sum

$$T = \sum_g \chi(f_g\rho_g)\chi.$$

In particular, T belongs to the corner $\chi(C_0(W)\rtimes_{\mathrm{red}}G)\chi$. Since this corner contains all the finite propagation operators on Y, we see that it contains $C^*(Y)$ as well.

The C^*-algebra $C_0(\partial W)\rtimes_{\mathrm{red}}G$ is handled by analogous computations, regarding χ as an element of $C_0(\partial W)$. Finally, under the identification of $C_0(G)\rtimes_{\mathrm{red}}G$ with $\mathcal{K}(\ell^2(G))$, it is clear that $\mathcal{K}(\ell^2(Y))=\chi(C_0(G)\rtimes_{\mathrm{red}}G)\chi$.

Having identified the C^*-algebras in the top row of the diagram with corners of those in the bottom row corresponding to the projection χ it remains to see that these corners are full. Again, we begin with the C^*-algebra $C_0(W) \rtimes_{\text{red}} G$. This crossed product is generated by operators of the form $f\rho_g$ where f is a bounded function with support in the set Yh, for some $h \in G$. Thus, it suffices to show that each such operator belongs to the ideal of $C_0(W) \rtimes_{\text{red}} G$ generated by χ. Now, the characteristic function of Yh, viewed as an operator on $\ell^2(G)$, is $\rho_h^* \chi \rho_h$. It follows that

$$f\rho_g = f(\rho_h^* \chi \rho_h)\rho_g = (f\rho_{h^{-1}})\chi\rho_{hg}$$

belongs to the ideal generated by χ, and we are through.

In a similar way, the image of χ is a full projection in $C_0(\partial W) \rtimes_{\text{red}} G$. Finally, any non-zero projection on $\ell^2(G)$, and in particular χ, is a full multiplier of $C_0(G) \rtimes_{\text{red}} G = \mathcal{K}(\ell^2(G))$. $\square$

Now, consider the diagram

$$\begin{tikzcd} K_0(\mathcal{K}) \arrow[r] \arrow[d] & K_0(C^*(Y)) \arrow[r] \arrow[d] & K_0(C^*_\infty(Y)) \arrow[d] \\ K_0(C_0(G) \rtimes_{\text{red}} G) \arrow[r] & K_0(C_0(W) \rtimes_{\text{red}} G) \arrow[r] & K_0(C_0(\partial W) \rtimes_{\text{red}} G) \end{tikzcd} \tag{7.3}$$

functorially induced by the diagram in the above lemma. We showed in Proposition 4.4 that the top line is not exact: the class of the *Kazhdan projection* in $K_0(C^*(Y))$ is not the image of a class from $K_0(\mathcal{K})$, but gets sent to zero in $K_0(C^*_\infty(Y))$. As the vertical maps are induced by inclusions of full corners, they are isomorphisms on K-theory, and so the bottom line is also not exact in the middle: again, the failure of exactness is detected by the class of the Kazhdan projection.

Unfortunately, we cannot appeal directly to Corollary 7.2 to show that Baum-Connes with coefficients fails for G, as the C^*-algebras $C_0(W)$ and $C_0(\partial W)$ are not separable. To get separable C^*-algebras with similar properties, let $C_0(Z)$ be any G-invariant C^*-subalgebra of $C_0(W)$ that contains $C_0(G)$; it follows that Z contains G as a dense open subset, and writing $\partial Z = Z \setminus G$ gives a short exact sequence of G-C^*-algebras.

$$0 \longrightarrow C_0(G) \longrightarrow C_0(Z) \longrightarrow C_0(\partial Z) \longrightarrow 0$$

We want to guarantee that the crossed product $C_0(Z) \rtimes_{\text{red}} G$ contains the Kazhdan projection. There is a straightforward way to do this: our efforts in this section culminate in the following theorem.

7.4. **Theorem.** *With notation as above, let $C_0(Z)$ denote any separable G-invariant C^*-subalgebra of $C_0(W)$ that contains $C_0(G)$ and the characteristic function χ of the expander Y. Then*

(i) the crossed product $C_0(Z) \rtimes_{red} G$ contains the Kazhdan projection associated to Y;
(ii) the sequence

$$K_0(C_0(G) \rtimes_{red} G) \longrightarrow K_0(C_0(Z) \rtimes_{red} G) \longrightarrow K_0(C_0(\partial Z) \rtimes_{red} G)$$

is not exact in the middle;
(iii) the Baum-Connes conjecture with coefficients is false for G.

Proof. For (i), let $d \in \mathbb{N}$ be the degree of all the vertices in Y. The Laplacian on Y (compare line (4.2) above) is then given by

$$\Delta = d\chi - \sum_{g \in G,\ |g|=1} \chi \rho_g \chi$$

and is thus in $C_0(Z) \rtimes_{\mathrm{red}} G$. As both Δ and χ are elements of $C_0(Z) \rtimes_{\mathrm{red}} G$, the Kazhdan projection p is as well, by the functional calculus.

Part (ii) follows from part (i), our discussion of $C_0(W)$ above, and the commutative diagram

$$\begin{array}{ccccc}
K_0(C_0(G) \rtimes_{\mathrm{red}} G) & \longrightarrow & K_0(C_0(Z) \rtimes_{\mathrm{red}} G) & \longrightarrow & K_0(C_0(\partial Z) \rtimes_{\mathrm{red}} G) \\
\| & & \downarrow & & \downarrow \\
K_0(C_0(G) \rtimes_{\mathrm{red}} G) & \longrightarrow & K_0(C_0(W) \rtimes_{\mathrm{red}} G) & \longrightarrow & K_0(C_0(\partial W) \rtimes_{\mathrm{red}} G)
\end{array},$$

where the vertical arrows are all induced by the canonical inclusions. Part (iii) is immediate from part (ii) and Corollary 7.2. □

At this point, we do not know exactly for which of the coefficients $C_0(Z)$ or $C_0(\partial Z)$ Baum-Connes fails. Indeed, the fact that Baum-Connes is true with coefficients in $C_0(G)$ and a chase of the diagram from Lemma 7.1 shows that either surjectivity fails for $C_0(Z)$, or injectivity fails for $C_0(\partial Z)$. A more detailed analysis in Theorem 9.7 below shows that in fact the assembly map is an isomorphism with coefficients in $C_0(\partial Z)$, so that surjectivity fails for G with coefficients in $C_0(Z)$.

8. Reformulating the conjecture: exotic crossed products

In this section, we discuss how to adapt the Baum-Connes conjecture to take the counterexamples from Section 7 into account. The counterexamples to the conjecture stem from analytic properties of the reduced crossed product: a natural way to adapt the conjecture is then to change the crossed product to one with 'better' properties.

Indeed, it is quite simple to define a 'conjecture of Baum-Connes type' for an arbitrary crossed product functor $\rtimes_\tau$. Define the *τ-Baum-Connes assembly map* to be the composition

$$K_*^{\mathrm{top}}(G;A) \to K_*(A \rtimes_{\max} G) \to K_*(A \rtimes_\tau G)$$

of the maximal assembly map, and the map induced on K-theory by the quotient map $A \rtimes_{\max} G \to A \rtimes_\tau G$; it follows from the discussion in Section 6 that this is the usual Baum-Connes assembly map when τ is the reduced crossed product. And one may hope that the τ-Baum-Connes assembly map is an isomorphism under favorable conditions for well behaved τ.

One certainly cannot expect all of these 'τ-Baum-Connes assembly maps' to be isomorphisms, however: indeed, we have already observed that isomorphism fails for some groups when τ is the reduced crossed product. There are even examples of a-T-menable groups and associated crossed products τ for which the τ-Baum-Connes assembly map (with trivial coefficients) is not an isomorphism: see [10, Appendix A]. Considering these examples as well as naturality issues, one is led to the following desirable properties of a crossed product functor τ that might be used to 'fix' the Baum-Connes conjecture.

Exactness. It should fix the exactness problems: that is, for any short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

of G-C^*-algebras, the induced sequence of C^*-algebras

$$0 \longrightarrow I \rtimes_\tau G \longrightarrow A \rtimes_\tau G \longrightarrow B \rtimes_\tau G \longrightarrow 0$$

should be exact.

Compatibility with Morita equivalences. Two G-C^*-algebras are *equivariantly stably isomorphic* if $A \otimes \mathcal{K}_G$ is equivariantly $*$-isomorphic to $B \otimes \mathcal{K}_G$. Here, $\mathcal{K}_G$ denotes the compact operators on the direct sum $\oplus_1^\infty \ell^2(G)$, equipped with the conjugation action arising from the direct sum of copies of the regular representation. It follows directly from the definition of E-theory that the domain of the Baum-Connes assembly map cannot detect the difference between equivariantly stably isomorphic coefficient algebras. Therefore we would like our crossed product to have the same property: see [10, Definition 3.2] for the precise condition we use.

This is a manifestation of Morita invariance. Indeed, separable G-C^*-algebras are equivariantly stably isomorphic if and only if they are equivariantly Morita equivalent, as follows from results in [23] and [53], which leads to a general Morita invariance result in E-theory [28, Theorem 6.12]. See also [18, Sections 4 and 7] for the relationship to other versions of Morita invariance.

Existence of descent. There should be a descent functor $\rtimes_\tau : \mathbf{E^G} \to \mathbf{E}$, which agrees with $\rtimes_\tau : \mathbf{GC}^* \to \mathbf{C}^*$ on G-C^*-algebras and $*$-homomorphisms. This is important for proving the conjecture: indeed, following the paradigm established by Kasparov [40], the most powerful known approaches to the Baum-Connes conjecture proceed by proving that certain identities hold in $\mathbf{E^G}$ (or in the KK^G-theory category, or some related more versatile setting as in Lafforgue's work [45]), and then using descent to deduce consequences for crossed products.

Consistency with property (T). The three properties above hold for the maximal crossed product. However, it is well-known that the maximal crossed product is not the right thing to use for the Baum-Connes conjecture: the *Kazhdan projections* (see [67] or [39, Section 3.7]) in $C^*_{\max}(G) = \mathbb{C} \rtimes_{\max} G$ are not in the image of the maximal assembly map $K_*^{\mathrm{top}}(G; \mathbb{C}) \to K_*(\mathbb{C} \rtimes_{\max} G)$ (see [32, Discussion below 5.1]). We would thus like that all Kazhdan projections get sent to zero under the quotient map $\mathbb{C} \rtimes_{\max} G \to \mathbb{C} \rtimes_\tau G$.

Summarizing the above discussion, any crossed product that 'fixes' the Baum-Connes conjecture should have the following properties:

(i) it is *exact*;
(ii) it is *Morita compatible*;
(iii) it has a *descent functor* in E-theory;
(iv) it annihilates *Kazhdan projections*.

Such crossed products do indeed exist! In order to prove this, we introduced in [10, Section 3] a partial order on crossed product functors by saying that $\rtimes_\tau \geq \rtimes_\sigma$ if the norm on $A \rtimes_{\mathrm{alg}} G$ coming from $A \rtimes_\tau G$ is at least as large as that coming from $A \rtimes_\sigma G$. The following theorem is one of the main results of [10]; the part dealing with exactness is due to Kirchberg.

8.1. **Theorem.** *With respect to the partial order above, there is a* (*unique*) *minimal crossed product* $\rtimes_{\mathcal{E}}$ *with properties* (*i*) *and* (*ii*). *This crossed product automatically also has properties* (*iii*) *and* (*iv*).

Summarizing, our reformulation of the Baum-Connes conjecture is that the *$\mathcal{E}$-Baum-Connes assembly* map

$$K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_{\max} G) \to K_*(A \rtimes_{\mathcal{E}} G)$$

is an isomorphism; we shall refer to this assertion as the *$\mathcal{E}$-Baum-Connes conjecture*.

It is quite natural to consider the minimal crossed product satisfying (i) and (ii) above: indeed, it is in some sense the 'closest' to the reduced crossed product among all the crossed products with properties (i) to (iv) above and, for exact groups it *is* the reduced crossed product. Consequently, for exact groups the reformulated conjecture is nothing other than the original Baum-Connes conjecture.

8.2. *Remark.* As mentioned above, we chose to work with E-theory, instead of the more common KK-theory to formulate the Baum-Connes conjecture. It is natural to ask whether the development above can be carried out in KK-theory, and in particular whether $\rtimes_{\mathcal{E}}$ admits a descent functor $\rtimes_{\mathcal{E}} : KK^G \to KK$. The answer is yes, as long as we restrict as usual to countable groups and separable G-C^*-algebras [18].

9. The counterexamples and the reformulated conjecture

In this section we shall revisit the counterexamples presented in Section 7 to the *original* Baum-Connes conjecture, and study them from the point of view of the *reformulated* conjecture of Section 8. In particular, we shall continue with the notation of Section 7: G is a Gromov monster group, containing an expander Y; $C_0(W)$ is the minimal G-invariant C^*-subalgebra of $\ell^\infty(G)$ that contains $\ell^\infty(Y)$; and $\partial W = W \setminus G$.

In Theorem 7.4 we saw that if $C_0(Z)$ is any separable G-invariant C^*-subalgebra of $C_0(W)$ containing both $C_0(G)$ and the characteristic function of Y, then the *original* Baum-Connes conjecture fails for G with coefficients in at least one of $C_0(Z)$ and $C_0(\partial Z)$, where again $\partial Z = Z \setminus G$. The key point was the failure of exactness of the sequence

$$K_0(C_0(G) \rtimes_{\mathrm{red}} G) \longrightarrow K_0(C_0(Z) \rtimes_{\mathrm{red}} G) \longrightarrow K_0(C_0(\partial Z) \rtimes_{\mathrm{red}} G)$$

in the middle, as evidenced by the Kazhdan projection of Y.

In the case of the *reformulated* conjecture, however, we would be working with the analogous sequence involving the $\mathcal{E}$-crossed product, which is exact (even at the level of C^*-algebras). Thus, at this point we know that the proof of Theorem 7.4 will not apply to show the existence of counterexamples to the $\mathcal{E}$-Baum-Connes conjecture. However, something much more interesting happens: in this section, we shall show that for any Z as above, the $\mathcal{E}$-Baum-Connes conjecture is true for G with coefficients in both $C_0(Z)$ and $C_0(\partial Z)$! This is a stronger result than in our original paper [10], where we just showed that there exists some Z with the property above.

There are two key-ingredients. First, we need a 'two out of three' lemma, which will allow us to deduce the $\mathcal{E}$-Baum-Connes conjecture for $C_0(Z)$ from the $\mathcal{E}$-Baum-Connes conjecture for $C_0(G)$ and $C_0(\partial Z)$. Second, we need to show that the action of G on ∂Z is always *a-T-menable*: this implies via work of Tu [65] that a strong form of the Baum-Connes conjecture holds in the equivariant E-theory category, and allows us to deduce the $\mathcal{E}$-Baum-Connes conjecture for G with coefficients in $C_0(\partial Z)$. The crucial geometric assumption needed for the second step is that the expander Y has large girth, and therefore looks 'locally like a tree'.

The first ingredient is summarized in the following lemma, which is a more precise version of Lemma 7.1. See [10, Proposition 4.6] for a proof. See also [19, Section 4] for a proof for the original formulation of the Baum-Connes conjecture using the reduced crossed product, and the additional assumption that G is exact on the level of K-theory.

9.1. **Lemma.** *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of separable G-C^*-algebras. There is a commutative diagram of six term sequences

$$\begin{tikzcd}
K_0^{top}(G;I) \arrow[rr] \arrow[dr] & & K_0^{top}(G;A) \arrow[rr] \arrow[dr] & & K_0^{top}(G;B) \arrow[dr] \arrow[dd] & \\
& K_0(I \rtimes_{\mathcal{E}} G) \arrow[rr] & & K_0(A \rtimes_{\mathcal{E}} G) \arrow[rr] & & K_0(B \rtimes_{\mathcal{E}} G) \arrow[dd] \\
K_1^{top}(G;B) \arrow[uu] \arrow[dr] & & K_1^{top}(G;A) \arrow[ll] \arrow[dr] & & K_1^{top}(G;I) \arrow[ll] \arrow[dr] & \\
& K_1(B \rtimes_{\mathcal{E}} G) \arrow[uu] & & K_1(A \rtimes_{\mathcal{E}} G) \arrow[ll] & & K_1(I \rtimes_{\mathcal{E}} G), \arrow[ll]
\end{tikzcd}$$

in which the front and back rectangular six term sequences are exact, and the maps from the back sequence to the front are $\mathcal{E}$-Baum-Connes assembly maps. In particular, if the Baum-Connes conjecture holds with coefficients in two out of three of I, A, and B, then it holds with coefficients in the third.

We now move on to the second key ingredient, the a-T-menability of the action of a Gromov monster group G on any of the spaces ∂Z.

9.2. **Definition.** Let G be a discrete group acting on the right on a locally compact space X by homeomorphisms. The action is *a-T-menable* if there is a continuous function

$$h : X \times G \to [0, \infty)$$

with the following properties.

(i) The restriction of h to $X \times \{e\}$ is 0.
(ii) For all $x \in X$ and $g \in G$, $h(x, g) = h(xg, g^{-1})$.
(iii) For any finite subset $\{g_1, ..., g_n\}$ of G, any finite subset $\{t_1, ..., t_n\}$ of $\mathbb{R}$ such that $\sum_{i=1}^{n} t_i = 0$, and any $x \in X$, we have that

$$\sum_{i,j=1}^{n} t_i t_j h(xg_i, g_i^{-1} g_j) \leq 0.$$

(iv) For any compact subset K of X, the restriction of h to the set

$$\{(x, g) \in X \times G \mid x \in K, xg \in K\}$$

is proper.

The following result is essentially [10, Theorem 7.9].

9.3. **Theorem.** *Let G be a Gromov monster group with isometrically embedded expander Y, and let W and ∂W be as in Section 7 (and as explained at the beginning of this section). Let $\pi : G \to Y$ be any function such that $d(x, \pi(x)) = d(x, Y)$ for all $x \in G$.*

Define a function $h : G \times G \to [0, \infty)$ by $h(x, g) = d(\pi(x), \pi(xg))$. Then h extends by continuity to a function $h : W \times G \to [0, \infty)$, and the restriction $h : \partial W \times G \to [0, \infty)$ has all the properties in Definition 9.2. In particular, the action of G on ∂W is a-T-menable.

The crucial geometric input into the proof is the fact that Y has large girth. This means that as one moves out to infinity in Y, then Y 'looks like a tree' on larger and larger sets. One can then use the negative type property of the distance function on a tree to prove that h has the right properties.

Now, let $C_0(Z)$ be any separable G-invariant subalgebra of $C_0(W)$ containing $C_0(G)$ and the characteristic function χ of Y. Set $\partial Z = Z \setminus G$. We would like to show that the action of G on ∂Z is also a-T-menable; as ∂Z is a quotient of ∂W, it suffices from Theorem 9.3 to show that the function $h : G \times G \to [0, \infty)$ extend to $h : Z \times G \to [0, \infty)$ (at least for some choice of function $\pi : G \to Y$ with the properties in the statement). We will do this via a series of lemmas.

9.4. **Lemma.** *For each $r > 0$, let $N_r(Y) = \{g \in G \mid d(g, Y) \leq r\}$ denote the r-neighborhood of Y in G. If $\overline{N_r(Y)}$ denotes the closure of $N_r(Y)$ in Z, then $\{\overline{N_r(Y)}\}_{r\in\mathbb{N}}$ is a cover of Z by an increasing sequence of compact, open subsets.*

Proof. For $g \in G$, let χ_{Yg} denote the characteristic function of the right translate of Y by g, which is in $C_0(Z)$ by definition of this algebra. Hence $f = \sum_{|g|\leq r} \chi_{Yg}$ is in $C_0(Z)$. The closure of $N_r(Y)$ is equal to $f^{-1}(0, \infty)$ and to $f^{(-1)}([1, \infty))$, and is thus compact and open as f is an element of $C_0(Z)$. Finally, note that finitely supported elements of $\ell^\infty(G)$ and translates of χ by the right action of G are supported in $N_r(Y)$ for some $r > 0$; as such elements generate Z, it follows that $\{\overline{N_r(Y)}\}_{r\in\mathbb{N}}$ is a cover of Z. □

Choose now an order $g_1, g_2, ...$ on the elements of G such that $g_1 = e$ and so that the function $\mathbb{N} \to \mathbb{R}$ defined by $n \mapsto |g_n|$ is non-decreasing. For each $x \in G$, let $n(x)$ be the smallest integer such that $xg_{n(x)}$ is in Y, and define a map $\pi : G \to Y$ by setting $\pi(x) = xg_{n(x)}$. Note that $d(\pi(x), x) = d(x, Y)$ for all $x \in G$.

9.5. **Lemma.** *Fix $g \in G$ and $r \in \mathbb{N}$, and define a function*

$$h_{r,g} : N_r(Y) \to [0, \infty),\ x \mapsto d(\pi(x), \pi(xg)).$$

Then $h_{r,g}$ extends continuously to the closure $\overline{N_r(Y)}$ of $N_r(Y)$ in Z.

Proof. Write the elements of $\{x \in G \mid |x| \leq r\}$ as $g_1, ..., g_n$ with respect to the order used to define π. For each $m \in \{1, ..., n\}$, let

$$E_m = \{x \in N_r(Y) \mid xg_m \in Y\}$$

and note that the characteristic function χ_{E_m} of E_m is equal to $\chi_{Yg_m} \cdot \chi_{N_r(Y)}$ and is thus in $C_0(Z)$ by Lemma 9.4. On the other hand, if we let

$$F_m = \{x \in N_r(Y) \mid \pi(x) = xg_m\}$$

then the characteristic function of F_m equals $\chi_{E_m}\big(1 - \sum_{i=1}^{m-1} \chi_{E_i}\big)$ and is thus also in $C_0(Z)$. Similarly, if we write the elements of $\{h \in G \mid |h| \leq r + |g|\}$ as $g_1, ..., g_{n'}$ and for each $m \in \{1, ..., n'\}$ let

$$F'_m = \{x \in N_r(Y) \mid \pi(xg) = xg_m\},$$

then the characteristic function of F'_m is in $C_0(Z)$.

For each $(k, l) \in \{1, ..., n\} \times \{1, ..., n'\}$, let $\chi_{k,l}$ denote the characteristic function of $F_k \cap F'_l$, which is in $C_0(Z)$ by the above discussion. Note that the restriction of $h_{r,g}$ to $F_k \cap F'_l$ sends $x \in N_r(Y)$ to

$$d(\pi(x), \pi(xg)) = d(xg_k, xgg_l) = |g_k^{-1} gg_l|.$$

Hence

$$h_{r,g} = \sum_{k=1}^{n} \sum_{l=1}^{n'} |g_k^{-1} g g_l| \chi_{k,l},$$

and thus $h_{r,g}$ is in $C_0(Z)$ as claimed. □

9.6. **Corollary.** *The function*

$$h : G \times G \to [0, \infty), \quad x \mapsto d(\pi(x), \pi(xg))$$

extends by continuity to $h : Z \times G \to [0, \infty)$. In particular, the action of G on ∂Z is a-T-menable.

Proof. For each fixed $g \in G$, Lemma 9.5 implies that the restriction of the function

$$h_g : G \to [0, \infty), \ x \mapsto d(\pi(x), \pi(xg))$$

to $N_r(Y)$ extends continuously to $\overline{N_r(Y)}$; as $\{\overline{N_r(Y)}\}_{r \in \mathbb{N}}$ is a compact, open cover of Z, it follows that h_g extends to a continuous function on all of Z. Hence the function

$$h : G \times G \to [0, \infty), \ (x, g) \mapsto d(\pi(x), \pi(xg))$$

extends to a continuous function on all of $Z \times G$. The result now follows from Theorem 9.3 as ∂Z is a quotient of ∂W. □

The following corollary is the culmination of our efforts in this section. First, it gives us more information about what goes wrong with the Baum-Connes conjecture than Section 7 does. More importantly for our current work, it shows that the $\mathcal{E}$-Baum-Connes conjecture is true for this counterexample: we thus have a concrete class of example where our reformulated conjecture 'out-performs' the original one.

9.7. **Theorem.** *Let G be a Gromov monster group with isometrically embedded expander Y. Equip $\ell^\infty(G)$ with the action induced by the right translation action of G on itself, and let $C_0(W)$ denote the G-invariant C^*-subalgebra of $\ell^\infty(G)$ generated by $\ell^\infty(Y)$. Let $C_0(Z)$ be any separable G-invariant C^*-subalgebra of $C_0(W)$ that contains $C_0(G)$ and the characteristic function χ of Y. Then:*

- *(i) the usual Baum-Connes assembly map for G with coefficients in $C_0(Z)$ is injective;*
- *(ii) the usual Baum-Connes assembly map for G with coefficients in $C_0(Z)$ fails to be surjective;*
- *(iii) the $\mathcal{E}$-Baum-Connes assembly map for G with coefficients in $C_0(Z)$ is an isomorphism.*

Proof. The essential point is that work of Tu [65] shows that a-T-menability of the action of G on ∂Z implies that a strong version of the Baum-Connes conjecture for G with coefficients in $C_0(\partial Z)$ holds in the equivariant E-theory category $\mathbf{E^G}$. This in turn implies the τ-Baum-Connes conjecture for G with coefficients in $C_0(\partial Z)$ for any crossed product τ that admits a descent functor. See [10, Theorem 6.2] for more details.

The result follows from this, Lemma 9.1, and the fact that the Baum-Connes conjecture is true for any crossed product with coefficients in a proper G-algebra like $C_0(G)$. $\square$

10. The Kadison-Kaplansky conjecture for $\ell^1(G)$

The Kadison-Kaplansky conjecture states that for a torsion free discrete group G, there are no idempotents in $C^*_{\mathrm{red}}(G)$ other than the 'trivial' examples given by 0 and 1. It is well-known that the usual Baum-Connes conjecture implies the Kadison-Kaplansky conjecture. As $\ell^1(G)$ is a subalgebra of $C^*_{\mathrm{red}}(G)$, the Kadison-Kaplansky conjecture implies that $\ell^1(G)$ contains no idempotents other than 0 or 1. In this section, we show that the $\mathcal{E}$-Baum-Connes conjecture, and in fact any 'exotic' Baum-Connes conjecture, implies that $\ell^1(G)$ has no non-trivial idempotents. Thus the $\mathcal{E}$-Baum-Connes conjecture implies a weak form of the Kadison-Kaplansky conjecture. Compare [14, Corollary 1.6] for a similar result in the context of the Bost conjecture.

10.1. **Theorem.** *Let G be a countable torsion free group and let σ be a crossed product functor for G. If the σ-Baum-Connes conjecture holds for G with trivial coefficients then the only idempotents in the Banach algebra $\ell^1(G)$ are zero and the identity.*

Recall that there is a canonical tracial state

$$\tau : C^*_{\mathrm{red}}(G) \to \mathbb{C}, \qquad \tau(a) = \langle\, \delta_e, a\delta_e \,\rangle.$$

The trace τ is well known to be faithful in the sense that a non-zero positive element of $C^*_{\mathrm{red}}(G)$ has *strictly* positive trace: see, for example [17, Proposition 2.5.3]. One has the following standard C^*-algebraic lemma.

10.2. **Lemma.** *Let $e \in C^*_{red}(G)$ be an idempotent. If $\tau(e)$ is an integer then $e = 0$ or $e = 1$.*

Proof. The idempotent e is similar in $C^*_{\mathrm{red}}(G)$ to a projection p [15, Proposition 4.6.2] so that $\tau(p) = \tau(e) \in \mathbb{Z}$. Positivity of τ and the operator inequality $0 \le p \le 1$ imply that $0 \le \tau(p) \le 1$ and so $\tau(p) = 0$ or $\tau(p) = 1$. Since τ is faithful, we conclude $p = 0$ or $p = 1$, and the same for e. $\square$

Recall that the trace τ defines a map $\tau_* : K_0(C^*_{\mathrm{red}}(G)) \to \mathbb{R}$ which sends the K-theory class of an idempotent $e \in C^*_{\mathrm{red}}(G)$ to $\tau(e)$ [15, Section 6.9]. The key point in the proof of Theorem 10.1 is the following result concerning the image under τ_* of elements in the range of the Baum-Connes assembly map.

10.3. **Proposition.** *Let G be a countable torsion free group. If $x \in K_0(C^*_{red}(G))$ is in the image of the Baum-Connes assembly map with trivial coefficients then $\tau_*(x)$ is an integer.*

Proof. This is a corollary of Atiyah's covering index theorem [6] (see also [20]). The most-straightforward way to connect Atiyah's covering index theorem to the Baum-Connes conjecture, and thus to prove the proposition, is via the Baum-Douglas geometric model for K-homology [9, 11]: this is explained in [69, Section 6.3] or [7, Proposition 6.1]. See also [48], particularly Theorem 0.3, for a slightly different approach (and a more general statement that takes into account the case when G has torsion). □

Proof of Theorem 10.1. Let e be an idempotent in $\ell^1(G)$. Since $\ell^1(G)$ is a subalgebra of *both* $C^*_{\mathrm{red}}(G)$ and $C^*_\sigma(G) := \mathbb{C} \rtimes_\sigma G$, we may consider the K-theory classes $[e]_{\mathrm{red}}$ and $[e]_\sigma$ defined by e for each of these C^*-algebras. The usual (reduced) Baum-Connes assembly map factors through the quotient map $C^*_\sigma(G) \to C^*_{\mathrm{red}}(G)$, and this quotient map is the identity on $\ell^1(G)$, so takes $[e]_\sigma$ to $[e]_{\mathrm{red}}$. Thus, since $[e]_\sigma$ is in the range of the σ-Baum-Connes assembly map, $[e]_{\mathrm{red}}$ is in the range of the reduced Baum-Connes assembly map. Proposition 10.3 implies now that $\tau(e)$ is an integer, and Lemma 10.2 implies that e is equal to either 0 or 1. □

11. Concluding remark

In our reformulated version of the Baum-Connes conjecture, the left side is unchanged, that is, is the same as in the original conjecture as stated by Baum and Connes [8]. At first glance, it may seem surprising that in the reformulated conjecture only the right hand side is changed. In this section we shall motivate, via the *Bost conjecture*, precisely why the left side should remain unchanged.

Recall that the original Baum-Connes assembly map for the group G with coefficients in a G-C^*-algebra A factors as

$$K_*^{\mathrm{top}}(G, A) \to K_*(\ell^1(G, A)) \to K_*(A \rtimes_{\mathrm{red}} G),$$

where $\ell^1(G, A)$ is the Banach algebra crossed product. According to the Bost conjecture, the first arrow in this display is an isomorphism; the second arrow is induced by the inclusion $\ell^1(G, A) \to A \rtimes_{\mathrm{red}} G$.

The Bost conjecture is known to hold in a great many cases, in particular, for fundamental groups of Riemannian locally symmetric spaces; see [45]. In these cases, the Baum-Connes conjecture is equivalent to the assertion that the K-theory of the Banach algebra $\ell^1(G, A)$ is isomorphic to the K-theory of the C^*-algebra $A \rtimes_{\text{red}} G$, and in fact that the inclusion $\ell^1(G, A) \to A \rtimes_{\text{red}} G$ induces an isomorphism.

While the Bost conjecture may seem more natural because it has the appropriate functoriality in the group G, it does not have the important implications for geometry and topology that the Baum-Connes conjecture does. In particular, it is not known to imply either the Novikov higher signature conjecture or the stable Gromov-Lawson-Rosenberg conjecture about existence of positive scalar curvature metrics on spin manifolds. As described here, our reformulation, which involves an appropriate C^*-algebra completion of $\ell^1(G, A)$ retains these implications.

From this point of view, an attempt to reformulate the Baum-Connes conjecture should involve finding a pre-C^*-norm on $\ell^1(G, A)$ with the property that the K-theory of the Banach algebra $\ell^1(G, A)$ equals the K-theory of its C^*-algebra completion. This problem involves in a fundamental way the harmonic analysis of the group, and this paper can be viewed as indicating a possible solution.

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