Coarse geometry of graph coverings

Erik Guentner

University of Hawaii at Manoa

(with Goulnara Arzhantseva, Jan Spakula)

Coarse embeddability

Z, a metric space

uniformly discrete, meaning $\inf d(x,y) > 0$ – bounded geometry, meaning $\forall r > 0$ the ball $B_r(x)$ has cardinality bounded independently of x – typically a graph or a group, or a coarse union of graphs or groups

ℓ^2, Hilbert space

$f : Z \to \ell^2$ is a coarse embedding $\Leftrightarrow$ there exist $\rho_\pm : \mathbb{R}^+ \to \mathbb{R}^+$, non-decreasing, proper s.t. for all x, $y \in Z$

$$\rho_-(d(x,y)) \leq \|f(x) - f(y)\| \leq \rho_+(d(,y))$$

Z is coarsely embeddable (CE) $\Leftrightarrow$ $\exists$ a coarse embedding

X can be drawn in $\mathcal{H}$ without excessive distortion – many natural examples coarse equivalence of metric spaces similarly defined – CE is a coarse property – in case of a group or graph, a coarse embedding is automatically Lipschitz

Coarse unions - box spaces

Coarse unions are interesting – they provide CnE examples, counterexamples to K-theoretic conjectures (Baum-Connes and Coarse Baum-Connes)

Z_i, finite (or bounded) metric spaces – the blocks

$\square = Z_1 \sqcup Z_2 \sqcup \dots$ (disjoint) union

$\square$ is a metric space; use any metric satisfying

(1) $Z_i \subset \square$ isometrically

(2) Z_i are well-spaced: $d(Z_i, Z_j) \to \infty$ as $i + j \to \infty$

(this is well-defined up to coarse equivalence)

Ex: the 'well-spaced line' is $\{ n^2 : n = 1, 2, \dots \}$

Ex: G a (residually finite) group; $\square G = \Gamma_0 \sqcup \Gamma_1 \sqcup \dots$ where

N_i normal, finite index subgroups, usually $\cap N_i = \{ 1 \}$, and $\Gamma_i = G/N_i$

Why box spaces?

The coarse geometry of a coarse union is essentially the 'uniform' (or equi?) coarse geometry of the blocks. In the case of the box space $\Box G$ this can sometimes be related to G itself – balls in G match up with balls in Γ_n for large n.

Ex: $\Box G$ is CE $\Leftrightarrow$ Γ_n are 'uniformly' CE $\Rightarrow$ G a-T-menable

cannot reverse this imiplication – it is not true that a quotient of an a-T-menable group is itself

Ex: $\Box G$ is coarsely amenable ($\equiv$ Property A) $\Leftrightarrow$ Γ_n are 'uniformly' coarsely amenable $\Leftrightarrow$ G is amenable

if interested, I will explain this at the end

Ex: $\Box G$ are expanders $\Leftrightarrow$ G has (τ) wrt the subgroups N_i

such box spaces are not CE – coarse unions of expanders are not CE

Expanders are not CE

Assume $\Box G = \sqcup \Gamma_n$ is CE. Equivalently, the Γ_n are 'uniformly' CE meaning $\exists \rho$ and $f_n : \Gamma_n \to \ell^2$ s.t. $\forall x, y \in \Gamma_n$

$$\rho(d(x,y)) \leq \|f(x) - f(y)\| \leq d(x,y).$$

The Γ_n are expanders means that $\exists\ \lambda > 0$ s.t. $\lambda_1(\Gamma_n) \geq \lambda$. That is, whenever $f : \Gamma_n \to \mathbb{R}$ and $\sum f(x) = 0$ then also

$$\lambda \sum_x \|f(x)\|^2 \leq \sum_{x \sim y} \|f(x) - f(y)\|^2.$$

Also true for ℓ^2-valued f. Thus,

$$\sum_x \|f(x)\|^2 \leq \lambda^{-1} \sum_{x \sim y} 1 \leq C\#(\Gamma_n), \qquad C = \lambda^{-1}\text{deg}/2$$

and at least half the vertices of Γ_n map to the ball of radius $\sqrt{2C}$. This contradicts existence of ρ, using bounded geometry – $\exists\ x_n,\ y_n \in \Gamma_n$ s.t. $d(x_n, y_n) \to \infty$ but $\|f_n(x_n) - f_n(y_n)\| \leq 2\sqrt{2C}$.

Questions

Does there exist a bounded geometry metric space that is CE and also coarsely non-amenable? Given the previous criterion, it is natural to look at spaces $\square G$ for non-amenable G – but not with Property T or (τ). Rephrase in terms of graphs:

Question: Does there exist a family of k-regular ($k \geq 3$) graphs Γ_n with girth tending to infinity which are 'uniformly' CE: $\exists \rho$ and $f_n : \Gamma_n \to \ell^2$ s.t. $\forall\, x,\, y \in \Gamma_n$

$$\rho(d(x,y)) \leq \|f_n(x) - f_n(y)\|_2 \leq d(x,y).$$

A related question in finite metric space geometry is (Linial, Magen, Naor):

Question: Does there exist a family as above having uniformly bounded ℓ^1-distortion: $\exists C$ and $f_n : \Gamma_n \to \ell^1$ s.t. $\forall\, x,\, y \in \Gamma_n$

$$C^{-1} d(x,y) \leq \|f_n(x) - f_n(y)\|_1 \leq d(x,y).$$

Questions

An answer to the second question answers the first question. Indeed, there exists $F : \ell^1 \to \ell^2$ for which

$$\|F(a) - F(b)\|_2 = \sqrt{\|a - b\|_1}$$

Define $F : \ell^1 \to \oplus L^2(\mathbb{R}) \cong \ell^2$ by: $a = (a_1, a_2, \dots) \in \ell^1$, $F(a) = (A_1, A_2, \dots)$ and $A_n =$ characteristic function of $[0, a_n]$ or $[a_n, 0]$ depending as a_n is positive or negative.

Now, compose.

Answers

Theorem (AGS): The answer to the first question is yes.

The above formulation was not available when we proved our theorem. We originally defined $\Box\mathbb{F}_2 = \Gamma_1 \sqcup \Gamma_2 \sqcup \ldots$ where $\Gamma_n = \mathbb{F}_2/N_n$ and the N_i are $N_0 = \mathbb{F}_2$, $N_1 = \mathbb{F}_2^{(2)}$, ...and, for example, $\mathbb{F}_2^{(2)}$ is the subgroup generated by the squares of elements – alternately, the blocks are the iterated $\mathbb{Z}/2$-homology covers of the 'figure 8'. The $\mathbb{Z}/2$-homology covers of graphs G_n with $\mathrm{girth}(G_n) \to \infty$ suffice.

Theorem (O): The answer to the second question is yes.

Ostrovskii followed our method – he defined the Γ_n to be the $\mathbb{Z}/2$-homology covers of graphs G_n for which $\mathrm{diam}(G_n) \leq \mathrm{girth}(G_n)$ and $\mathrm{girth}(G_n) \to \infty$ and gave a more detailed analysis of the metric on Γ_n.

Wall spaces and cuts

S a set

$W = \{ A, B \}$ a wall – a decomposition $S = A \sqcup B$, A, B nonempty

W separates x and y if $x \in A$ and $y \in B$ or the other way around

$\mathcal{W}$ a collection of walls with the property:

for every x, y only finitely many W separate

Prop: $d(x, y) =$ the number of walls separating x and y defines a metric on S; with this metric S embeds isometrically into ℓ^1 and is CE with $\rho_\pm(r) = \sqrt{r}$.

define $f : S \to \ell^2(\mathcal{W})$ by $f(x) =$ characteristic function of those walls separating x from a fixed basepoint

When Γ is a graph, we speak of cuts: a collection of edges with the property that when they are removed the resulting graph has exactly two connected components. The components define a wall.

Covering spaces

The blocks in our example are the iterated $\mathbb{Z}/2$-homology covers of the 'figure-8'

Γ a graph, $\pi_1(\Gamma) \cong \mathbb{F}_r$

$\mathbb{F}_r \to \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ (r-summands)

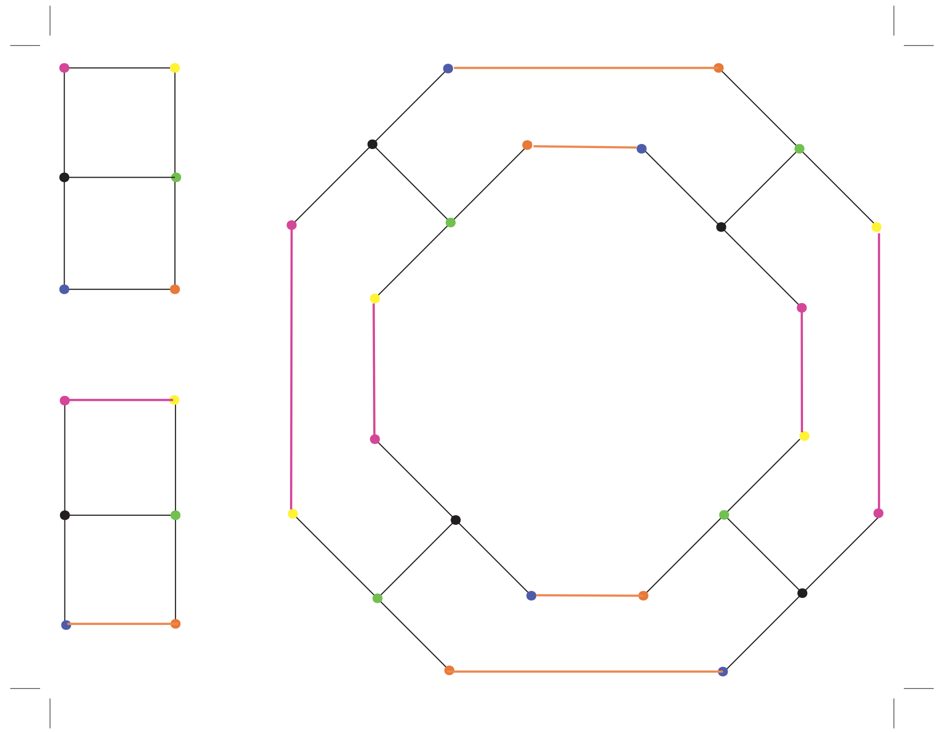
the $\mathbb{Z}/2$-homology cover $\widetilde{\Gamma}$ of Γ is the corresponding cover

- its fundamental group is the kernel of this homomorphism
- it admits a simple geometric construction

Construction - example

given graph Γ
$\pi_1(\Gamma) \cong \mathbb{F}_2$

maximal tree in black
red and orange edges
are generators of π_1



the cover – 4 copies of the tree, placed as vertices of $\mathbb{Z}/2 \times \mathbb{Z}/2$,
with red and orange edges connecting these ‘clouds’

Cuts in covers

Lemma: If Γ has the property that every edge belongs to a circuit then the edges in $\widetilde{\Gamma}$ sitting over a given edge in Γ form a cut

So, $\widetilde{\Gamma}$ has a wall structure in which walls correspond to edges in Γ

Observe: The Γ_n in our tower are not expanders.

The ‘cubical’ cuts give an inequality involving Cheeger constants: $h(\widetilde{\Gamma}) \leq 2/\#(\Gamma)$. Recall,

$$h(\Gamma) = \inf \frac{\#E(A,B)}{\min\{\#(A), \#(B)\}}.$$

When the cardinality of the base tends to infinity, the Cheeger constant tends to zero.

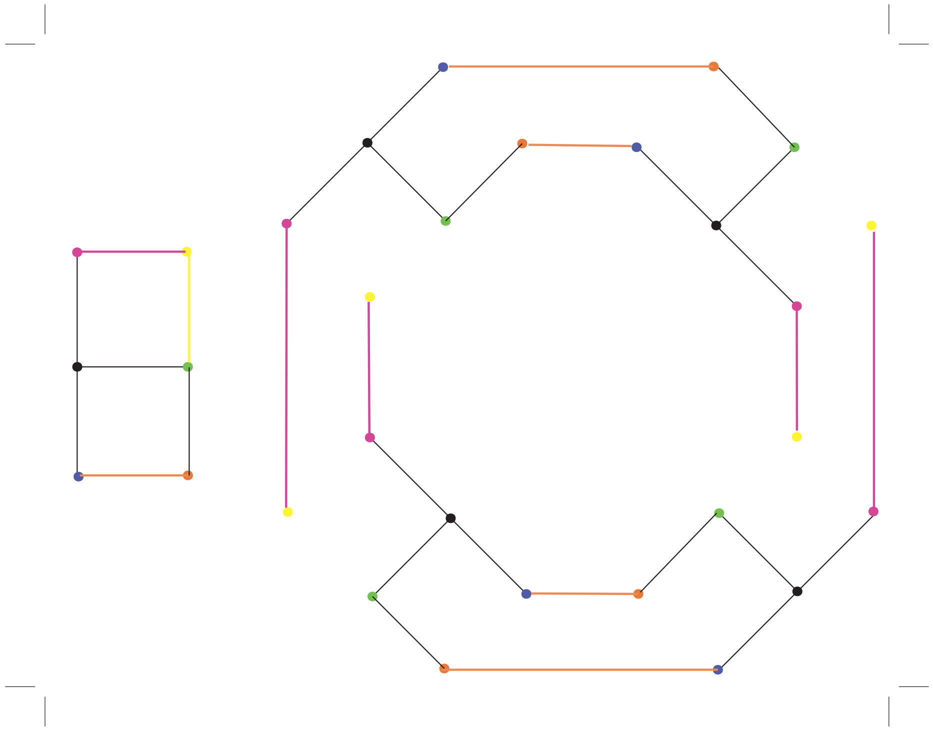
In our tower, each block now has a wall metric. These are isometrically embedded into ℓ^1 and 'uniformly' CE. This would mean that $\Box\mathbb{F}_2$ (but with coarse union of wall metrics) is CE.

Cuts - example

it is clear from construction that the red edges form a cut in the cover
it is less obvious that the edges sitting over an edge in the maximal tree form a cut in the cover

an edge in the maximal tree, now yellow

edges sitting over the yellow edge form a cut in the cover



Covering spaces

But, on $\widetilde{\Gamma}$ the wall and graph metrics do not coincide.

Neither in general, nor on our example.

Prop: The wall and graph metrics on $\widetilde{\Gamma}$ coincide on scales smaller than the girth of Γ. Precisely, the wall distance and graph distance between two points in $\widetilde{\Gamma}$ agree when one (equivalently both) of them is smaller than the girth of Γ.

This shows that $\square\mathbb{F}_2$ with either metric is simultaneously CE – also simultaneously CA. Since it is CnA with graph metrics and CE with wall metrics we are done.

The main tool is unique path lifting for covers.

Slightly more refined analysis gives the Lipschitz lower bound in Ostrovskii's result.

Coarse amenability $\equiv$ Property A

Z a metric space

uniformly discrete and bounded geometry – typically a graph or a group, or a coarse union of graphs or groups – just insurance

Z is coarsely amenable (CA) $\Leftrightarrow$ for every $R > 0$, $\varepsilon > 0$ there exists $S > 0$ and $f : Z \to \mathbf{Prob}(Z)$ s.t.

(1) if $f_z(x) \neq 0$ then $d(x, z) \leq S$

(2) if $d(x, y) \leq R$ then $\|f_x - f_y\| \leq \varepsilon$

$Z = G$ a (countable, discrete) group

CA is a non-equivariant formulation of amenability – the definition is analogous to the Reiter condition

CA is equivalent to topological amenability of the action of G on its Stone-Cech compactification.

Examples and counterexamples

Most naturally occurring spaces and groups are CA.

(1) free groups, hyperbolic groups
(2) amenable groups
(3) linear groups (not necessarily discrete)
(4) mapping class groups
(5) symmetric spaces and buildings
(6) CAT(0) cube complexes (finite dimensional)
(7) CA (and CE, too) closed under many operations

Since CA implies CE, these are CE as well.

Expander graphs are not CE, hence also not CA.

graphs Z_n with $\lambda_1(Z_n) \geq \lambda > 0$ cannot 'uniformly' coarsely embed