Graph coverings, coarse non-amenability and coarse embeddings

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Coarse amenability $\equiv$ Property A

Z a metric space

uniformly discrete and bounded geometry – typically a graph or a group, or a coarse union of graphs or groups – just insurance

Z is coarsely amenable (CA) $\Leftrightarrow$ for every $R > 0$, $\varepsilon > 0$ there exists $S > 0$ and $f : Z \to \mathbf{Prob}(Z)$ s.t.

(1) if $f_z(x) \neq 0$ then $d(x, y) \leq S$

(2) if $d(x, y) \leq R$ then $\|f_x - f_y\| \leq \varepsilon$

$Z = G$ a (countable, discrete) group

CA is a non-equivariant formulation of amenability – the definition is analogous to the Reiter condition

CA is equivalent to topological amenability of the action of G on its Stone-Cech compactification – compare Sageev's talk

Coarse embeddability

Z a metric space

$\mathcal{H}$, Hilbert space

$f : Z \to \mathcal{H}$ is a coarse embedding $\Leftrightarrow$ there exist $\rho_\pm : \mathbb{R}^+ \to \mathbb{R}^+$, non-decreasing, proper s.t. for all x, $y \in Z$

$$\rho_-(d(x,y)) \leq \|f(x) - f(y)\| \leq \rho_+(d(,y))$$

Z is coarsely embeddable (CE) $\Leftrightarrow$ $\exists$ a uniform embedding

X can be drawn in $\mathcal{H}$ without excessive distortion

$Z = G$ a group

CE is a non-equivariant formulation of a-T-menability – an equivariant coarse embedding is the orbit of an affine isometric action

Examples and counterexamples

Most naturally occurring spaces and groups are CA.

(1) free groups, hyperbolic groups
(2) amenable groups
(3) linear groups (not necessarily discrete)
(4) mapping class groups
(5) symmetric spaces and buildings
(6) CAT(0) cube complexes (finite dimensional)
(7) CA (and CE, too) closed under many operations

Since CA implies CE, these are CE as well.

Expander graphs are not CE, hence also not CA.

graphs Z_n with $\lambda_1(Z_n) \geq \lambda > 0$ cannot 'uniformly' coarsely embed

Does there exist a space or group that is CE, but CnA?

Coarse unions - box spaces

Z_i finite (or bounded) metric spaces

$\Box = Z_1 \sqcup Z_2 \sqcup \ldots$ (disjoint) union

$\Box$ is a metric space; use any metric satisfying

(1) $Z_i \subset \Box$ isometrically

(2) Z_i are well-spaced: $d(Z_i, Z_j) \to \infty$ as $i + j \to \infty$

(this is well-defined up to coarse equivalence)

G a (residually finite) group

$\Box G = \Gamma_0 \sqcup \Gamma_1 \sqcup \ldots$ where

N_i normal, finite index subgroups, usually $\cap N_i = \{ 1 \}$, and $\Gamma_i = G/N_i$

Interesting class of spaces – CnE examples, counterexamples to K-theoretic conjectures (Baum-Connes and Coarse Baum-Connes)

Our example

Theorem (AGS): There exists a $\Box\mathbb{F}_2$ that is CE and CnA. This example has bounded geometry.

> (at the level of equivariant properties, $\mathbb{F}_2$ is a-T-menable but not amenable, so the question is natural)

Construction: $\Box\mathbb{F}_2 = \Gamma_0 \sqcup \Gamma_1 \sqcup \ldots$ where N_i are defined by $N_0 = \mathbb{F}_2$, $N_1 = \mathbb{F}_2^{(2)}, \ldots$ and, for example, $\mathbb{F}_2^{(2)}$ is the subgroup generated by the squares of elements

Theorem (N): The space $\Box = \mathbb{Z}/2 \sqcup \mathbb{Z}/2 \times \mathbb{Z}/2 \sqcup \ldots$ is CE and CnA; the metrics on the blocks are the Hamming distance.

> but this space does not have bounded geometry – further, unbounded geometry is used essentially in the proof of CnA

Why box spaces?

Fact: $\Box G$ is CA $\Leftrightarrow G$ amenable

Fact: $\Box G$ is CE $\Rightarrow G$ a-T-menable

Fact: $\Box G$ are expanders $\Leftrightarrow G$ has (τ) wrt the subgroups N_i

such box spaces are neither CE nor CA

Idea: a box space is CA precisely when its blocks are 'uniformly' CA – when the blocks are (finite) groups you can average to see this happens precisely when the blocks are 'uniformly' amenable – in the case of $\Box G$, assuming the tower is faithful, this implies G itself is amenable because balls in G match up with balls in Γ_n for large n; it follows even when the tower is not faithful

This only works one way in the case of CE – it is not true that a quotient of an a-T-menable group is itself a-T-menable

So, our $\Box\mathbb{F}_2$ is CnA. What about CE?

Wall spaces and cuts

S a set

$W = \{ A, B \}$ a wall – a decomposition $S = A \sqcup B$, A, B nonempty

W separates x and y if $x \in A$ and $y \in B$ or the other way around

$\mathcal{W}$ a collection of walls with the property:

for every x, y only finitely many W separate

Prop: $d(x, y) =$ the number of walls separating x and y defines a metric on S; with this metric S is CE and one may take $\rho_{\pm}(r) = \sqrt{r}$.

define $f : S \to \ell^2(\mathcal{W})$ by $f(x) =$ characteristic function of those walls separating x from a fixed basepoint

When Γ is a graph, we speak of cuts: a collection of edges with the property that when they are removed the resulting graph has exactly two connected components. The components define a wall.

Covering spaces

The blocks in $\square\mathbb{F}_2$ are the $\mathbb{Z}/2$-homology covers of the 'figure-8'

Γ a graph, $\pi_1(\Gamma) \cong \mathbb{F}_r$

$\mathbb{F}_r \to \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$ (r-summands)

the $\mathbb{Z}/2$-homology cover $\widetilde{\Gamma}$ of Γ is the corresponding cover

its fundamental group is the kernel of this homomorphism
it admits a simple geometric construction

Lemma: If Γ has the property that every edge belongs to a circuit then the edges in $\widetilde{\Gamma}$ sitting over a given edge in Γ form a cut

So, $\widetilde{\Gamma}$ has a wall structure in which walls correspond to edges in Γ

In our tower, each block now has a wall metric and these are 'uniformly' CE. This would mean that $\square\mathbb{F}_2$ (but with coarse union of wall metrics) is CE.

Covering spaces

But, on $\widetilde{\Gamma}$ the wall and graph metrics do not coincide.

Neither in general, nor on our tower.

Prop: The wall and graph metrics on $\widetilde{\Gamma}$ coincide on scales smaller than the girth of Γ. Precisely, the wall distance and graph distance between two points in $\widetilde{\Gamma}$ agree when one (equivalently both) of them is smaller than the girth of Γ.

This shows that $\Box\mathbb{F}_2$ with either metric is simultaneously CE – also simultaneously CA. Since it is CnA with graph metrics and CE with wall metrics we are done.