A *perfect square* is a number a such that $a = b^2$ for some real number b. Some examples of perfect squares are $4 = 2^2, 16 = 4^2, 169 = 13^2$. We wish to have a method for finding b when a is an expression. For instance, you should remember that $a^2 + 2ab + b^2$ is a perfect square, because it is exactly $(a+b)^2$. How would you turn the expression $x^2 + ax$ into a perfect square?

A moment of thought should convince you that if we add $(\frac{a}{2})^2$ to $x^2 + ax$ we obtain a perfect square, because $(x + \frac{a}{2})^2 = x^2 + ax + (\frac{a}{2})^2$. The addition of $(\frac{a}{2})^2$ is called *completing the square*, because the new expression can now be written as a square of some other expression.

Example 1. Complete the square: $x^2 + 4x = 0$

$$x^2 + 4x = 0 \iff (x^2 + 4x + 4) = 4 \iff (x+2)^2 = 4$$

We have added *the square of half the coefficient of* x to the original equation, and therefore to maintain equality it was necessary to add the same amount to the other side of the equation.

Warning 2. The coefficient of x^2 must be equal to 1 in order to complete the square.

Example 3. Complete the square: $2x^2 + 8x = 0$

$$2x^2 + 8x = 0 \iff 2(x^2 + 4x) = 0 \iff 2(x^2 + 4x + 4) = 8 \iff 2(x+2)^2 = 8$$

We added 4, the square of half the coefficient of x, inside the parentheses. Note that this amounts to adding 8 to the left side of the equation, because everything inside the parentheses is multiplied by 2. Therefore, to maintain equality we add 8 to the right side of the equation. In case we cannot set our expression equal to 0, we must subtract whatever number we add to the expression:

Example 4. Complete the square: $2x^2 + 8x$

$$2x^2 + 8x = 2(x^2 + 4x) = 2(x^2 + 4x + 4) - 8 = 2(x+2)^2 - 8$$

Example 5. $(x-h)^2 + (y-k)^2 = r^2$ is the equation of a circle of radius r centered at the point (h,k). Using the method of completing the square (twice) find the radius and center of the circle given by the equation $x^2 + y^2 + 8x - 6y + 21 = 0$.

$$\begin{aligned} x^2 + y^2 + 8x - 6y + 21 &= 0 && (1) \\ (x^2 + 8x) + (y^2 - 6y) &= -21 && (2) \\ (x^2 + 8x + 16) + (y^2 - 6y + 9) &= -21 + 16 + 9 && (3) \\ (x+4)^2 + (y-3)^2 &= 4 && (4) \end{aligned}$$

We have now the form $(x - (-4))^2 + (y-3)^2 = 2^2$ which is a circle of radius $r = 2$ centered at the point $(h,k) = (-4, 3)$.

Deriving the Quadratic Formula Given a quadratic equation, i.e. an equation of this form:

$$ax^2 + bx + c = 0, a \neq 0 \tag{5}$$

where a, b, and c are real numbers, we wish to have a formula that will give us the explicit values of x for which the quadratic equation is zero. That is, we need a formula that produces x_1 and x_2 such that

$$ax_1^2 + bx_2 + c = 0 \text{ and } ax_2^2 + bx_2 + c = 0 \tag{6}$$

The quadratic formula tells us exactly how to find our set of solutions $\{x_1, x_2\}$, but it also tells how large this set is. We can have two distinct solutions and this happens whenever the discriminant is a positive number. We can have just one solution if the discriminant is zero. In this case we say that the root $x_1 (= x_2)$ has multiplicity 2, because it occurs twice. Finally, when the discriminant is a negative number, we have a square root of a negative number and hence no (real) solutions. Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ where the discriminant is equal to } b^2 - 4ac \tag{7}$$

How do we know that this is indeed correct? We can apply the method of completing the square to our quadratic equation (1) and verify that equation (2) is correct. Here are the details:

$$\begin{aligned}
ax^2 + bx + c &= 0 && (8)\\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && (9)\\
x^2 + \frac{b}{a}x &= -\frac{c}{a} && (10)\\
x^2 + \frac{b}{a}x + \left(\frac{\frac{b}{a}}{2}\right)^2 &= \left(\frac{\frac{b}{a}}{2}\right)^2 - \frac{c}{a} && (11)\\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} && (12)\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} && (13)\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a} && (14)\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2} && (15)\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} && (16)\\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} && (17)\\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} && (18)\\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} && (19)
\end{aligned}$$

$$x = -\frac{b}{2a} + \frac{\pm\sqrt{b^2-4ac}}{2a} \tag{20}$$
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \tag{21}$$

So which of the above steps do we call "completing the square"? The answer is (4) to (7); the rest deal with writing the equation in the form $x = \ something$. Let's review:

Suppose you are given your favorite quadratic $ax^2 + bx + c$ and need to solve for x. You are no longer amused by factoring and decide to complete the square instead.
Step 1: Check the coefficients. If $a = 0$ you don't need to complete the square. If $a \neq 1$ then you need to factor out a. So suppose that $a \neq 1$ and $a \neq 0$.

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] \tag{22}$$

Step 2: Group the x terms together. You complete the square only on the terms containing the variable x. Notice that inside the brackets [ ] we now have a new quadratic equation with coefficients $a = 1, b = \frac{b}{a}$ and $c = \frac{c}{a}$.

$$ax^2 + bx + c = a\left[\left(x^2 + \frac{b}{a}x\right) + \frac{c}{a}\right] \tag{23}$$

Step 3: Complete the square: add the square of half of the coefficient of x to the terms in side the parentheses ( ).

$$ax^2 + bx + c = a\left[\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + \frac{c}{a}\right] \tag{24}$$

Step 4: Up until now we have not altered the equation, but adding something to the right side requires subtracting the same number. We have added $\frac{b^2}{4a^2}$ inside the brackets [ ] and everything inside [ ] is multiplies by a. Therefore, to keep the equation unchanged, we now subtract from the right side the number $a \cdot \frac{b^2}{4a^2}$ and obtain

$$ax^2 + bx + c = a\left[\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + \frac{c}{a}\right] - \frac{b^2}{4a} \tag{25}$$

Step 5: Simplify. The term in the parentheses ( ) is a perfect square and so

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a}\right] - \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a} \tag{26}$$

This form should look familiar. If we were to set line (22) equal to zero we would have the standard quadratic equation. Then dividing by a (legal since $a \neq 0$) and moving terms around returns us to equation (12).

Viete's Equations, or how to pick out the correct pair of solutions to a quadratic equation ...

Proposition 6. *Given a quadratic equation with real coefficients* a, b, c

$$ax^2 + bx + c = 0, a \neq 0$$

If the solutions exist, then they have the following form

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and they obey the following algebraic equations:

$$x_1 + x_2 = \frac{-b}{a}$$

$$x_1 \cdot x_2 = \frac{c}{a}$$

If you are given a quadratic equation to solve *and* are allowed to use the quadratic formula, then you may follow these steps and save yourself some work.

Step 1: Make sure that the solutions exist, i.e. $b^2 - 4ac \geq 0$

Step 2: Look at the quadratic equation you have to solve and determine the values of a, b, c and compute $\frac{-b}{a}$ and $\frac{c}{a}$.

Step 3: Compute $x_1 + x_2$ and $x_1 \cdot x_2$ for each set of solutions your are given as a choice.

Step 4: Compare the results of steps 2 and 3. If you find a match, you have found the solution. If there is no match, then none of the possible choices is a solution. (Is it possible to have more than one set of matching solutions?)

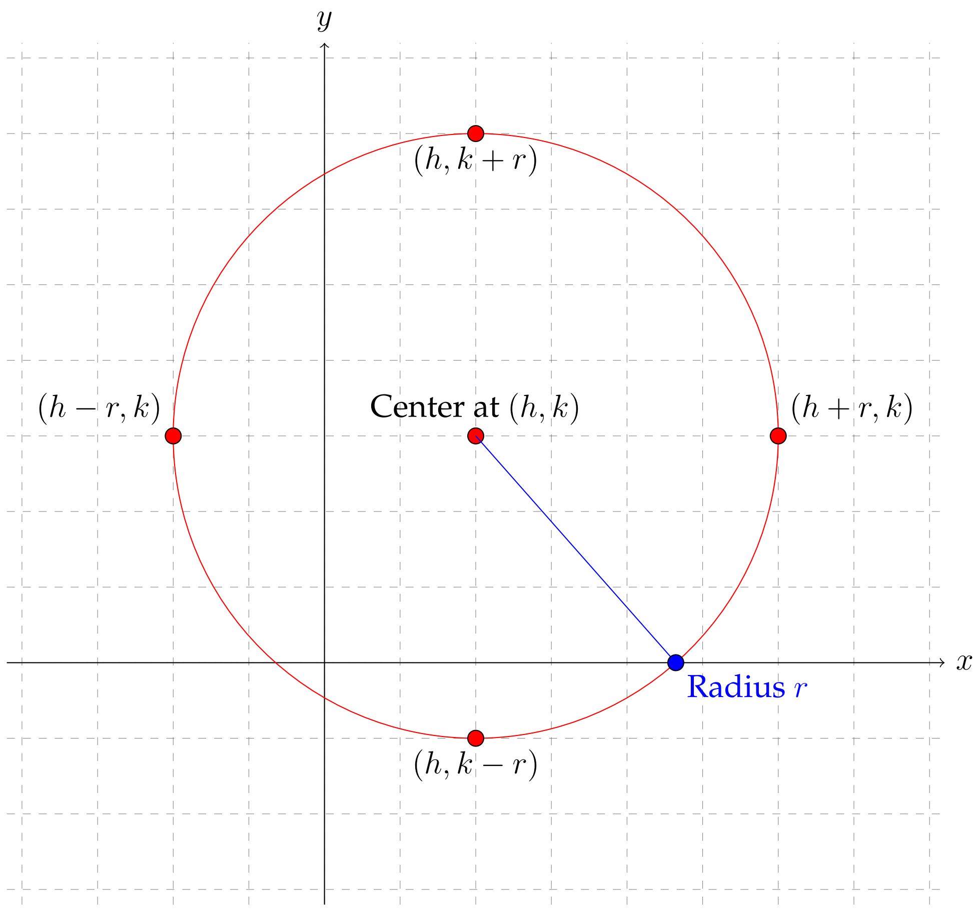
Using the method of completing the square, put each circle into the form

$$(x-h)^2 + (y-k)^2 = r^2$$

.

Then determine the center and radius of each circle.

1. $x^2 + y^2 - 10x + 2y + 17 = 0.$
2. $x^2 + y^2 + 8x - 6y + 16 = 0.$
3. $9x^2 + 54x + 9y^2 - 18y + 64 = 0.$
4. $4x^2 - 4x + 4y^2 - 59 = 0.$



Sample Midterm					Sample Final
3	A	B	C	D	
8	A	B	C	D	
11	A	B	C	D	
29	A	B	C	D	
31	A	B	C	D	
32	A	B	C	D	
35	A	B	C	D	