Remember these important identities! For all real numbers $a, b$:

**Difference of Squares:** $a^2 - b^2 = (a + b)(a - b)$

**Difference of Cubes:** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

**Sum of Cubes:** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$(a + b)^2 = (a^2 + 2ab + b^2)$

$(a + b)^3 = (a^3 + 3a^2b + 3ab^2 + b^3)$

**WARNING! Avoid the ”Freshman’s Dream”**

Remember to multiply all the factors in expressions such as (4) and (5) above. In general:

$$(a + b)^n \neq a^n + b^n$$

We will be using the difference of squares extensively. For example, how would you simplify the following expression so that the denominator no longer contains a square root?

$$\frac{5}{7 + \sqrt{x}}$$

The key idea is to realize that $(7 + \sqrt{x})(7 - \sqrt{x}) = 49 - x$, i.e. we have a difference of squares. Then if we multiply our equation by $1 = \frac{7 - \sqrt{x}}{7 - \sqrt{x}}$, we are not changing the equation, but the denominator will no longer have a radical.

$$\frac{5}{7 + \sqrt{x}} \cdot \frac{7 - \sqrt{x}}{7 - \sqrt{x}} = \frac{5(7 - \sqrt{x})}{49 - x} = \frac{35 - 5\sqrt{x}}{49 - x}$$

Remember that $a$ and $b$ in the difference of squares expression are parameters. This means that we can replace $a$ and $b$ with pretty much any expression we wish. Consider the following example. We are still using the difference of squares.

$$-\frac{(x - 2)^2}{2\sqrt{x - 1} - x} = -\frac{(x - 2)^2}{2\sqrt{x - 1} - x} \cdot \frac{2\sqrt{x - 1} + x}{2\sqrt{x - 1} + x}$$

$$= -\frac{(x - 2)^2 \cdot (2\sqrt{x - 1} + x)}{4(x - 1) - x^2}$$

$$= -\frac{(x - 2)^2 \cdot (2\sqrt{x - 1} + x)}{x^2 - 4x + 4}$$

$$= -\frac{(x - 2)^2 \cdot (2\sqrt{x - 1} + x)}{(x - 2)^2}$$

$$= 2\sqrt{x - 1} + x$$

In the above examples both the pairs $7 - \sqrt{x}$ and $7 + \sqrt{x}$ and the pairs $2\sqrt{x - 1} - x$ and $2\sqrt{x - 1} + x$ are called *conjugates*. In the difference of squares formula $(a + b)$ is the conjugate of $(a - b)$. So conjugation amounts to switching the sign in the given expression. The process of multiplying an expression whose denominator contains a radical by 1 in the form of a fraction with the numerator and denominator both being conjugates of the expression’s denominator is called *rationalizing the denominator*. 
Example 1. Rationalize the Denominator.

\[
\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5}}{\sqrt{3}} \cdot \left( \frac{\sqrt{3}}{\sqrt{3}} \right) = \frac{\sqrt{15}}{3}
\]

Example 2. Rationalize the denominator.

\[
\frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{\sqrt{4}} \cdot \left( \frac{\sqrt{4}}{\sqrt{4}} \right) = \frac{\sqrt{8}}{4} = \frac{\sqrt{2} \cdot \sqrt{2^{3}}}{4} = \frac{\sqrt{2} \cdot 2^{3}}{2}
\]

Example 3. Find the conjugate of \(\sqrt{x^2 - \frac{x}{2} - 2 + x}\).

\[
\sqrt{x^2 - \frac{x}{2} - 2 - x}
\]

Example 4. Rationalize the denominator.

\[
\frac{3x + 4}{2\sqrt{x^2 - \frac{x}{2} - 2 + x}} = \frac{3x + 4}{2\sqrt{x^2 - \frac{x}{2} - 2 + x}} \cdot \left( \frac{2\sqrt{x^2 - \frac{x}{2} - 2 + x}}{2\sqrt{x^2 - \frac{x}{2} + 2 - x}} \right)
\]

\[
= \frac{(3x + 4)(2\sqrt{x^2 - \frac{x}{2} - 2} - x)}{4(x^2 - \frac{x}{2} + 2) - x^2}
\]

\[
= \frac{(3x + 4)(2\sqrt{x^2 - \frac{x}{2} - 2} - x)}{3x^2 - 2x + 8}
\]

\[
= \frac{(3x + 4)(2\sqrt{x^2 - \frac{x}{2} - 2} - x)}{(3x + 4)(x - 2)}
\]

\[
= \frac{2\sqrt{x^2 - \frac{x}{2} - 2} - x}{x - 2}
\]
Rationalize the denominator:

1. \( \frac{\sqrt{2}}{\sqrt{3}} \)

2. \( \frac{\sqrt{2}}{1-\sqrt{2}} \)

3. \( \frac{1+\sqrt{3}}{1-\sqrt{3}} \)

4. \( \frac{a+\sqrt{b}}{a-\sqrt{b}} \)

5. \( \frac{1}{\sqrt{5}} + 4\sqrt{45} \)

6. \( \frac{4}{\sqrt{16}} \)

7. \( \frac{3}{\sqrt{3}} \)

8. \( \frac{3}{\sqrt[4]{27a^b}} \)

9. \( \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}} \)

10. \( \frac{\sqrt{x-2\sqrt{y}}}{\sqrt{x+2\sqrt{y}}} \)