A NOTE ABOUT THE GEOMETRIC OPTIMAL CONTROL OF THE COPEPOD SWIMMER

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1. Introduction. Swimming microorganisms employ a variety of mechanisms of propulsion, and they have inspired numerous models starting with undulating sheets and filaments introduced in the fifties [17, 9]. Recent studies have explored optimal strategies for swimming with minimal amount of mechanical work, an important criterion for assessing the fitness of different organisms and for designing efficient...
robotic swimmers [10]. Previous studies have computed optimal solutions in the
framework of variational analysis of optimal control[1],[3],[16],[4],[6].

A well-studied model is the Purcell swimmer [14], which consists of three rigid
and slender rods representing respectively the leg, the body, and the arm. The
configuration of the swimmer is described by two angles \( \theta = (\theta_1, \theta_2) \) with three
other variables \( q = (x, y, \Phi) \) describing respectively the position and the orientation
of the body. The system can be written as

\[
\begin{align*}
\dot{q} &= D(\Phi)G(\theta)\dot{\theta}, \\
\dot{\theta} &= H(\theta)\tau,
\end{align*}
\]

where \( D(\Phi) \) is the rotation matrix

\[
D(\Phi) = \begin{pmatrix} 
\cos(\Phi) & -\sin(\Phi) & 0 \\
\sin(\Phi) & \cos(\Phi) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

By denoting \( u = \dot{\theta} \), the mechanical power is

\[
\tau u = uH^{-1}u
\]

and the energy minimization problem becomes

\[
\int_0^T (uH^{-1}u)dt
\]

where \( u \) is taken as the control variable (see [12] for a complete description
of the matrices \( G \) and \( H \)). This control problem falls into the Sub-Riemannian (SR)
framework. It is complex even locally and is related to the Cartan flat SR-model
[4].

It is helpful to consider a simpler SR-case, the simplest being the Heisenberg
case. The Heisenberg control system is given by

\[
\begin{align*}
\dot{x} &= u_1\theta_2 - u_2\theta_1 \\
\dot{\theta}_1 &= u_1, \\
\dot{\theta}_2 &= u_2
\end{align*}
\]

while minimizing the integral

\[
\int_0^T (u_1^2 + u_2^2)dt.
\]

A simple integration of the non planar geodesics starting from the origin gives

\[
\begin{align*}
\theta_1(t) &= \frac{A}{\lambda} (\sin(\lambda t + \varphi) - \sin(\varphi)) \\
\theta_2(t) &= \frac{A}{\lambda} (\cos(\lambda t + \varphi) - \cos(\varphi)) \\
x(t) &= \frac{A^2}{\lambda} t - \frac{A^2}{\lambda^2} \sin(\lambda t)
\end{align*}
\]

where \( A, \lambda \) and \( \varphi \) are parameters related to the initial velocity. Both angular
variables are periodic motion corresponding to strokes to provide the displacement
\( x(2\pi/\lambda) \), whose average is given by \( A^2/\lambda^2 \), the period of the stroke being \( 2\pi/\lambda \).
A standard computation of conjugate points shows that such geodesic is optimal
up to \( t = 2\pi/\lambda \) (included). In this example, the optimal displacement is shown to
correspond to a simple stroke: indeed the first first conjugate time is at \( t = 2\pi/\lambda \)
and moreover it corresponds to the cut time.
While this model can provide some insights on optimal locomotion, it is too primitive because:
1. The geodesic flow is integrable due to a symmetry of revolution along $Ox$ and every $\theta$-motion is periodic.
2. The model is quasi-homogeneous, $\theta_1$ and $\theta_2$ are of weight 1 and $x$ is of weight 2, and invariant in the Heisenberg group.

Recently a new model was developed to mimic the locomotion of larval copepods, an abundant type of zooplankton thriving in the ocean [15, 13]. The simplest form of the model, hereafter referred to as the copepod swimmer, consists of a symmetric body consisting of two pairs of legs, with the first pair making an angle $\theta_1$ and the second pair making an angle $\theta_2$ with respect to the displacement direction $Ox$ (Fig.1).

The swimming velocity at $x_0$ is given by
\[
\dot{x}_0 = \frac{\dot{\theta}_1 \sin(\theta_1) + \dot{\theta}_2 \sin(\theta_2)}{2 + \sin^2(\theta_1) + \sin^2(\theta_2)}
\]
and the controls are the angular velocities
\[
\dot{\theta}_1 = u_1, \quad \dot{\theta}_2 = u_2.
\]
We also have the state constraints $\theta_i \in [0, \pi], \ i = 1, 2, \ \theta_1 \leq \theta_2$.

A simplified cost can be identified as
\[
\int_0^T (u_1^2 + u_2^2) dt
\]
but the true cost corresponding to the mechanical energy of the system is given by the quadratic form
\[
\dot{q}^T M \dot{q},
\]
where $q = (x_0, \theta_1, \theta_2)$, and $M$ is the symmetric matrix
\[
M = \begin{pmatrix}
2 - 1/2(\cos^2(\theta_1) + \cos^2(\theta_2)) & -1/2 \sin(\theta_1) & -1/2 \sin(\theta_2) \\
-1/2 \sin(\theta_1) & 1/3 & 0 \\
-1/2 \sin(\theta_2) & 0 & 1/3
\end{pmatrix}.
\]

This copepod swimmer serves as a suitable model for computing optimal controls in the framework of SR-geometry. The system is three-dimensional (two controls and one variable), which is arguably simpler than the five-dimensional system (two controls and three variables) of the previously studied Purcell swimmer. It is a global model of SR-geometry which can be analyzed in detail, showing in particular
2. Preliminary results.

2.1. **Geometric analysis of a copepod swimmer.**

A (general) stroke consists of periodic motion in the angular variables $\theta_1, \theta_2$. The period has no influence on the resultant displacement of the swimmer so it is fixed to $2\pi$ without loss of generality. The displacement $x_0(2\pi) - x_0(0)$ can be computed for any prescribed motion of $\theta_1, \theta_2$.

In [15], two types of geometric motions are described:

**First case:** (Fig. 2) The two legs are assumed to oscillate sinusoidally according to

$$\theta_1 = \Phi_1 + a \cos(t), \quad \theta_2 = \Phi_2 + a \cos(t + k_2)$$

with $a = \pi/4, \Phi_1 = \pi/4, \Phi_2 = 3\pi/4$ and $k_2 = \pi/2$. This produces a displacement $x_0(2\pi) = 0.2$. 

In accordance with the observations (see [2] for a similar model but a different analysis in which second order optimality conditions are not used). In addition, the optimal controls could be compared with observations of copepods to determine whether they are optimizing their strokes to minimize energy. Copepods must swim in order to find food and escape from predators, and they have had a chance to adapt and evolve over millions of years, but it remains unknown to what extent they have adapted their strokes to maximize their swimming efficiency. Thus the model optimization could offer new insight into biological behavior.

This note is organized in two sections. In section 2, we recall some properties of the copepod swimmer [15] and the mathematical tools from geometric optimal control (see [5] for a general reference). Section 3 contains the contribution of this note based on a geometric analysis and numerical simulations to describe the optimal strokes.
Second case: (Fig.3) The two legs are paddling in sequence followed by a recovery stroke performed in unison. In this case the controls \( u_1 = \dot{\theta}_1, u_2 = \dot{\theta}_2 \) produce bang arcs to steer the angles between from the boundary 0 of the domain to the boundary \( \pi \), while the unison sequence corresponds to a displacement from \( \pi \) to 0 with the constraint \( \theta_1 = \theta_2 \).

![Figure 3. Two legs paddling in sequence. The legs perform power strokes in sequence and then a recovery stroke in unison, each stroke sweeping an angle \( \pi \).](image)

Our main objective is to relate these properties to geometric optimal control.

2.2. Abnormal curves in the copepod swimmer.
We introduce \( q = (x_0, \theta_1, \theta_2) \), then the system is written as a driftless affine control system

\[
\dot{q}(t) = \sum_{i=1}^{2} u_i(t) F_i(q(t))
\]

where the control vector fields are given by

\[
F_i = \frac{\sin(\theta_i)}{\Delta} \frac{\partial}{\partial x_0} + \frac{\partial}{\partial \theta_i}
\]

with \( \Delta = 2 + \sin^2(\theta_1) + \sin^2(\theta_2) \). We denote by \( D \) the distribution generated by the two vector fields: \( D = \text{span} \{ F_1, F_2 \} \).

The Lie bracket of two vector fields \( F, G \) is computed with the convention

\[
[F, G](q) = \frac{\partial F}{\partial q}(q)G(q) - \frac{\partial G}{\partial q}(q)F(q).
\]

Finally, we denote by \( p = (p_1, p_2, p_3) \) the adjoint vector associated with \( q \).
We first recall a basic fact concerning the local classification of two-dimensional distributions, in relation with abnormal curves.
2.2.1. Local classification of two-dimensional distributions in dimension three and abnormal curves.

Let \( D = \text{span} \{ G_1, G_2 \} \) be the distribution generated by two vector fields \( G_1, G_2 \) in \( \mathbb{R}^3 \). Let \( z = (q, p) \) and denote \( H_i(z) = \langle p, G_i(q) \rangle, \ i = 1, 2 \) the Hamiltonian lifts. The Poisson bracket is given by

\[
\{ H_1, H_2 \}(z) = dH_1(\vec{H}_2)(z) = \langle p, [G_1, G_2](q) \rangle.
\]

Abnormal curves are defined by

\[
H_1(z) = H_2(z) = 0,
\]

and differentiating using the dynamics

\[
\frac{dz}{dt} = \sum_{i=1}^{2} u_i \vec{H}_i(z)
\]

we obtain the relations

\[
\{ H_1, H_2 \}(z) = 0 \quad u_1 \{ \{ H_1, H_2 \}, H_1 \}(z) + u_2 \{ \{ H_1, H_2 \}, H_2 \}(z) = 0
\]

defining the corresponding abnormal controls.

Tools from singularity theory can be used to classify the distributions, see [18]. Here we present only the two (stable) models related to our study.

**Contact case.** We say that \( q_0 \) is a contact point if \( \{ G_1, G_2, [G_1, G_2] \} \) is of dimension 3 at \( q_0 \). At a contact point, identified to 0, there exists a system of local coordinates \( q = (x, y, z) \) such that

\[
D = \ker(\alpha), \quad \alpha = ydx + dz.
\]

Observe that \( \alpha = dy \wedge dx \) (Darboux form) and that \( \frac{\partial}{\partial z} \) is the characteristic direction of \( \alpha \). This form is equivalent to

\[
D = \ker(\alpha'), \quad \alpha' = dz + (xdy - ydx)
\]

with

\[
D = \text{span} \{ G_1, G_2 \}, \quad G_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad G_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \quad G_3 = [G_1, G_2] = 2 \frac{\partial}{\partial z}.
\] (2)

**The Martinet case.** A point \( q_0 \) is a Martinet point if at \( q_0 \), \( [G_1, G_2] \in \text{span} \{ G_1, G_2 \} \) and at least one Lie bracket \( [[G_1, G_2], G_1] \) or \( [[G_1, G_2], G_2] \) does not belong to \( D \). Then, there exist local coordinates \( q = (x, y, z) \) near \( q_0 \) identified to 0 such that

\[
D = \ker(\omega), \quad \omega = dz - \frac{y^2}{2} dx
\]

where

\[
G_1 = \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial z}, \quad G_2 = \frac{\partial}{\partial y}, \quad G_3 = [G_1, G_2] = y \frac{\partial}{\partial z}
\] (3)

\[
[[G_1, G_2], G_1] = 0, \quad [[G_1, G_2], G_2] = \frac{\partial}{\partial z}.
\]

The surface \( \Sigma : y = 0 \) is called the Martinet surface and is foliated by abnormal curves, solutions of \( \frac{\partial}{\partial z} \). In particular, through the origin it corresponds to the curve \( t \rightarrow (t, 0, 0) \).
Remark 1. One more generic situation is the so-called tangential case, which will be not needed in our analysis.

2.2.2. Computations in the copepod case.
We have
\[ F_3 = [F_1, F_2] = f(\theta_1, \theta_2) \frac{\partial}{\partial x_0}, \]
with
\[ f(\theta_1, \theta_2) = \frac{2 \sin(\theta_1) \sin(\theta_2)(\cos(\theta_1) - \cos(\theta_2))}{\Delta^2}. \]

In particular we deduce the following lemma.

**Lemma 2.1.** The singular set \( \Sigma : \{ q; \det(F_1(q), F_2(q), [F_1, F_2](q)) = 0 \} \), where the vector fields \( F_1, F_2, [F_1, F_2] \) are collinear, is given by

- \( \theta_1 = 0 \) or \( \pi \),
- \( \theta_2 = 0 \) or \( \pi \),
- \( \theta_1 = \theta_2 \).

It is formed by the boundary of the physical domain: \( \theta_i \in [0, \pi], \theta_1 \leq \theta_2 \), with respective controls \( u_1 = 0, \ u_2 = 0 \) or \( u_1 = u_2 \).

**Remark 2.** The previous lemma provides the interpretation of the policy represented in Fig.3. In the shape space \((\theta_1, \theta_2)\) it corresponds to a triangle.

To analyse the first situation of Fig.2, the mechanical energy has to be used in relation with SR-geometry.

2.3. Sub-Riemannian geometry.
The problem is written
\[ \dot{q} = \sum_{i=1}^{2} u_i G_i(q), \quad \min_{u_i} \int_0^T (u_1^2 + u_2^2) dt, \]
where the cost is defined for a fixed final time \( T \) and corresponds to the energy. In our case we use the normalized value \( T = 2\pi \). In this representation, we assume that the vector fields \( G_1, G_2 \) are orthonormal.

According to Pontryagin maximum principle, we introduce the pseudo-Hamiltonian in the normal case
\[ H(z, u) = \sum_{i=1}^{2} u_i H_i(z) - \frac{1}{2} \sum_{i=1}^{2} u_i^2, \]
where \( H_i \) are the Hamiltonian lifts \( \langle p, G_i(q) \rangle \). The maximization condition is equivalent to \( \frac{\partial H}{\partial u_i} = 0, \ i = 1, 2 \). It follows that \( u_i = H_i \) and plugging this expression for \( u_i \) into \( H \) produces the real Hamiltonian in the normal case
\[ H_n = \frac{1}{2} (H_1^2 + H_2^2) . \]

In the contact situation, it corresponds to the Heisenberg case while in the Martinet situation, it corresponds to the flat Martinet case. In both cases it amounts to
impose that $G_1, G_2$ are orthonormal and that the associated distribution is nilpotent.

**Definition 2.2.** A normal stroke is a solution of $H_n$ such that $\theta_1$ and $\theta_2$ are periodic with period $2\pi$.

According to the transversality conditions of the maximum principle the dual variables $p_2$ and $p_3$ are such that $p_2$ and $p_3$ are both periodic of period $2\pi$ (to produce a smooth solution).

**Second order optimality condition.** In the normal case, the first conjugate point corresponds to the first point where a normal geodesic ceases to be minimizing which respect to the $C^1$-topology on the set of curves and they can be computed using the Hampath code [7].

This leads to the following definition.

**Definition 2.3.** A normal stroke is called optimal on $[0, 2\pi]$ if there exists no conjugate point on the interval $]0, 2\pi]$.

3. Computations and analysis in the copepod swimmer.

3.1. Simplified cost.

We start by considering the simplified cost

$$\min_{u(.)} \int_0^T (u_1^2 + u_2^2)dt$$

in relation with the contact case.

Outside the singular set $\Sigma$, we have only contact points. Introducing the Hamiltonian lifts: $H_i = \langle p, F_i(q) \rangle$ for $i = 1, 2$, and $H_3 = \langle p, [F_1, F_2](q) \rangle$, the set $\{q, H_1, H_2, H_3\}$ are coordinates and we have

$$H_1 = \frac{p_1 \sin \theta_1}{\Delta} + p_2, \quad H_2 = \frac{p_1 \sin \theta_2}{\Delta} + p_3, \quad H_3 = \frac{2p_1 \sin \theta_1 \sin \theta_2 \cos \theta_1 - \cos \theta_2}{\Delta^2}.$$  \hfill (4)

Moreover the problem is isoperimetric since $x_0$ is a cyclic coordinate, i.e. $p_1$ is a first integral: $\dot{p}_1 = 0$.

Straightforward computations lead to the expressions

$$\dot{H}_1 = dH_1(\vec{H}_n) = dH_1 \left( \frac{1}{2} (H_1^2 + H_2^2) \right) = \{H_1, H_2\} H_2,$$

$$\dot{H}_2 = dH_2(\vec{H}_n) = dH_1 \left( \frac{1}{2} (H_1^2 + H_2^2) \right) = \{H_2, H_1\} H_1.$$

This can be expressed in the following condensed way:

$$\dot{H}_1 = H_2 H_3, \quad \dot{H}_2 = -H_1 H_3.$$

Moreover,

$$\dot{H}_3 = dH_3(\vec{H}_n) = dH_3 \left( \frac{1}{2} (H_1^2 + H_2^2) \right) = \{H_3, H_1\} H_1 + \{H_3, H_2\} H_2$$

with

$$\{H_3, H_1\}(z) = \langle p, [[F_1, F_2], F_1](q) \rangle, \quad \{H_3, H_2\}(z) = \langle p, [[F_1, F_2], F_2](q) \rangle.$$
At a contact point \( \{F_1, F_2, F_3\} \) forms a frame, therefore
\[
[[F_1, F_2], F_1](q) = \sum_{i=1}^{3} \lambda_i(q) F_i(q)
\]
and computing one gets,
\[
\lambda_1 = \lambda_2 = 0, \quad \frac{\partial f}{\partial \theta_1} = \lambda_3 f.
\]
Similarly,
\[
[[F_1, F_2], F_2](q) = \sum_{i=1}^{3} \lambda'_i(q) F_i(q),
\]
with
\[
\lambda'_1 = \lambda'_2 = 0, \quad \frac{\partial f}{\partial \theta_2} = \lambda'_3 f.
\]
We conclude that
\[
\dot{H}_1 = H_2 H_3, \quad \dot{H}_2 = -H_1 H_3, \quad \dot{H}_3 = H_3 (\lambda_3 H_1 + \lambda'_3 H_2).
\]
We introduce a new time reparametrization with \( ds = H_3 dt \), and we obtain
\[
\frac{dH_1}{ds} = H_2, \quad \frac{dH_2}{ds} = -H_1, \quad \frac{dH_3}{ds} = \lambda_3 H_1 + \lambda'_3 H_2.
\]
Hence we have the harmonic oscillator since \( H''_1 + H_1 = 0 \) when differentiating with respect to the new time \( s \).

Furthermore \( H_3 \) can be computed using the remaining equation (5). Observe that with the approximation \( \lambda_3, \lambda'_3 \) constant, the equation is
\[
\frac{dH_3}{ds} = A \cos(s + \rho).
\]
In this formalism, the Heisenberg case corresponds to \( \lambda_3 = \lambda'_3 = 0 \).

3.2. True Cost.
In this case the computation is more intricated since \( F_1, F_2 \) are not orthonormal. Using optimal control formalism and the fact that \( \theta_1 = u_1, \theta_2 = u_2 \), the energy associated with the mechanical energy matrix \( M \) defined by (1) is written as
\[
a(q)u_1^2 + 2b(q)u_1 u_2 + c(q)u_2^2,
\]
where
\[
a = \frac{1}{3} - \frac{\sin^2 \theta_1}{2(2 + \sin^2 \theta_1 + \sin^2 \theta_2)},
\]
\[
b = -\frac{\sin \theta_1 \sin \theta_2}{2(2 + \sin^2 \theta_1 + \sin^2 \theta_2)},
\]
\[
c = \frac{1}{3} - \frac{\sin^2 \theta_2}{2(2 + \sin^2 \theta_1 + \sin^2 \theta_2)}.
\]
The pseudo-Hamiltonian in the normal case becomes
\[
H(q, p) = u_1 H_1(q, p) + u_2 H_2(q, p) - \frac{1}{2} \left( a(q)u_1^2 + 2b(q)u_1 u_2 + c(q)u_2^2 \right).
\]
and the normal controls are computed solving the equations

\[
\frac{\partial H}{\partial u_1} = 0, \quad \frac{\partial H}{\partial u_2} = 0.
\]

From an easy calculation, we obtain

\[
\begin{align*}
  u_1 &= -\frac{3(4H_1 + 2H_1 \sin^2 \theta_1 + 3H_2 \sin \theta_1 \sin \theta_2 - H_1 \sin^2 \theta_2)}{\sin^2 \theta_1 + \sin^2 \theta_2 - 4}, \\
  u_2 &= -\frac{9H_1 \sin \theta_1 \sin \theta_2 + 6H_2(2 + \sin^2 \theta_2) - 3H_2 \sin^2 \theta_1}{\sin^2 \theta_1 + \sin^2 \theta_2 - 4}.
\end{align*}
\]

Plugging such \( u \) into the pseudo-Hamiltonian gives the true Hamiltonian

\[
H_n = u_1 H_1 + u_2 H_2 - \frac{1}{2} \left( a(q) u_1^2 + 2b(q) u_1 u_2 + c(q) u_2^2 \right)
\]

\[
= -\frac{3}{2(\sin^2 \theta_1 + \sin^2 \theta_2 - 4)(\sin^2 \theta_1 + \sin^2 \theta_2 + 2)} \left( 2(\sin^2 \theta_1 + \sin^2 \theta_2) p_1^2 \\
+ (2 \sin^4 \theta_1 + \sin^2 \theta_1 \sin^2 \theta_2 - \sin^4 \theta_2 + 8 \sin^2 \theta_1 + 2 \sin^2 \theta_2 + 8) p_2^2 \\
+ (- \sin^4 \theta_1 + \sin^2 \theta_1 \sin^2 \theta_2 + 2 \sin^4 \theta_2 + 2 \sin^2 \theta_1 + 8 \sin^2 \theta_2 + 8) p_3^2 \\
+ (4 \sin^3 \theta_1 + 4 \sin^2 \theta_2 \sin \theta_1 + 8 \sin \theta_1) p_1 p_2 \\
+ (4 \sin^2 \theta_1 \sin \theta_2 + 4 \sin^3 \theta_2 + 8 \sin \theta_2) p_1 p_3 \\
+ (6 \sin \theta_2 \sin^3 \theta_1 + 6 \sin^3 \theta_2 \sin \theta_1 + 12 \sin \theta_1 \sin \theta_2) p_2 p_3 \right)
\]  

and leads to the dynamics in the \((q,p)\) coordinates

\[
\begin{align*}
  \dot{q}_1 &= \frac{\partial H_n}{\partial p_1} = -3(4 \sin \theta_1 \sin \theta_2 p_1 + (8 - 2 \sin^2 \theta_1 + 4 \sin^2 \theta_2) p_2 + 6 \sin \theta_1 \sin \theta_2 p_1 p_2) \\
  \dot{q}_2 &= \frac{\partial H_n}{\partial p_2} = -3(2 \sin \theta_1 p_1 + 3 \sin \theta_1 \sin \theta_2 p_2 + (4 - \sin^2 \theta_1 + 2 \sin^2 \theta_2) p_3) \\
  \dot{q}_3 &= \frac{\partial H_n}{\partial p_3} = -3(2 \sin \theta_1 p_1 + 3 \sin \theta_1 \sin \theta_2 p_2 + (4 - \sin^2 \theta_1 + 2 \sin^2 \theta_2) p_3) \\
\end{align*}
\]
We use the Hampath code [7] at two levels:

3.3. Numerical computations and justification of the policy of Fig.2.

We use the Hampath code [7] at two levels:

1. The shooting equations associated with the problem are

\[ x_0(0) = 0, \quad x_0(2\pi) = x_f, \]
\[ \theta_{1,2}(0) = \theta_{1,2}(2\pi), \quad p_{2,3}(0) = p_{2,3}(2\pi). \]

where for the simulations we set \( x_f = 0.2 \) to be compared with Fig.2.

State variables, adjoint variables and control variables are illustrated in Fig.4-5 for the simplified cost \( \int (u_1^2 + u_2^2) \) and in Fig.6-7 for the mechanical (true) cost.

In Fig.8-9 and Fig.10-11, these solutions are extended to \( t > 2\pi \) so that sequences of three identical strokes are represented.

2. The Hampath code is also used to show that the normal stroke is optimal testing the nonexistence of conjugate points using the variational equation to compute Jacobi fields. Recall that according to [5], given a reference curve \((q(t), p(t))\) solution in the normal case, a time \( t_c \in [0, T] \) is a conjugate time if there exists a Jacobi field \( \delta z = (\delta q, \delta p) \), that is a non-zero solution of the variational equation

\[ \delta z(t) = \frac{\partial H_n}{\partial z}(q(t), p(t)) \delta z(t) \] (7)
such that $\delta q(0) = \delta q(t_c) = 0$. We denote $\delta z_i = (\delta q_i, \delta p_i)$, $i = 1...n$, $n$-independent solutions of (7) with initial condition $\delta q(0) = 0$. At time $t_c$ we have the following rank condition

$$\text{rank}\{\delta q_1(t_c), ..., \delta q_n(t_c)\} < n.\quad (8)$$

Fig.12 (resp. Fig.13) represents the smallest singular value associated with the rank condition (8) for the $\int (u_1^2 + u_2^2)$ cost (resp. mechanical cost). We conclude the nonexistence of conjugate time for both costs.

4. Conclusion.
The objective of this note is to point the interest of the copepod swimmer which can be analyzed in the framework of SR-geometry showing the accordance of the observation with simple computations concerning both normal and abnormal solutions. Further studies will allow a complete understanding of the model. A more complete analysis is clearly based on the following: understand the role of the boundary of the domain: $\theta_1 \in [0, \pi]$, $\theta_2 \in [0, \pi]$, $\theta_2 \geq \theta_1$ corresponding respectively to abnormal curves associated with amplitude bounds on the strokes. In particular connections between normal and abnormal curves is related to the calculation of the optimal solutions on the bounded domain using the maximum principle but with state constraints.

Complementary figures 15-20 are finally presented as a first step toward the computation of the global solution. For simplicity the state constraints have not be taken into account. They represent various stroke shapes: non self-intersecting case, the eight case and the limaçon. In all cases we have represented conjugate point location on a stroke (if existing) and the energy level corresponding to the cost of the stroke. See also Fig.17 for more self intersections.
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Figure 5. Time evolution of the adjoint variables $p = (p_1, p_2, p_3)$ and the control variables $u = (u_1, u_2)$ for the simplified cost $\int (u_1^2 + u_2^2)$.

Figure 6. Time evolution of the state variables $q = (x_0, \theta_1, \theta_2)$ for the mechanical cost.
Figure 7. Time evolution of the adjoint variables $p = (p_1, p_2, p_3)$ and the control variables $u = (u_1, u_2)$ for the mechanical cost.

Figure 8. A sequence of three identical strokes for the simplified cost $f(u_1^2 + u_2^2)$ (state variables).
Figure 9. A sequence of three identical strokes for the simplified $\int (u_1^2 + u_2^2)$ (adjoint variables and control variables).

Figure 10. A sequence of three identical strokes for the mechanical cost (state variables).
Figure 11. A sequence of three identical strokes for the mechanical cost (adjoint variables and control variables).

Figure 12. Second order sufficient condition checked on a stroke for the simplified cost $\int (u_1^2 + u_2^2)$. The smallest singular value associated with the rank condition (8) doesn’t vanish on $[0, 2\pi]$, and there is no conjugate time.
Figure 13. Second order sufficient condition checked on a stroke for the mechanical cost. The smallest singular value associated with the rank condition (8) doesn’t vanish on \([0, 2\pi]\), and there is no conjugate time.

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Figure 14. Optimal stroke for the $\int (u_1^2 + u_2^2)$ cost: non intersecting case. We fixed the displacement to $x_0(2\pi) = 0.22$.

Figure 15. Optimal stroke for the $\int (u_1^2 + u_2^2)$ cost: limacon with inner loop. We fixed the displacement to $x_0(2\pi) = 0.22$.

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Figure 16. Optimal stroke for the $\int (u_1^2 + u_2^2)$ cost: eight case. We fixed the displacement to $x_0(2\pi) = 0.22$.

Figure 17. Optimal stroke for the $\int (u_1^2 + u_2^2)$ cost: multiple self intersections. We fixed the displacement to $x_0(2\pi) = 0.22$.

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Figure 18. Optimal stroke for the mechanical cost: non intersecting case. We fixed the displacement to $x_0(2\pi) = 0.25$.

Figure 19. Optimal stroke for the mechanical cost: limaçon with inner loop case. We fixed the displacement to $x_0(2\pi) = 0.25$. 
Figure 20. Optimal stroke for the mechanical cost:eight case. We fixed the displacement to $x_0(2\pi) = 0.25$. 