

Local Approximation with Kernels

Thomas Hangelbroek

University of Hawaii at Manoa

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A cubic spline example

Consider breakpoints $t_0 = 0 < t_1 \dots < t_n = 1$ with local mesh ratio

$$M = \max_{|i-j| \leq 1} \frac{t_{i+1} - t_i}{t_{j+1} - t_j}.$$

Marsden ('74): For $M > (3 + \sqrt{5})/2$, there are knot sequences so that (periodic) cubic spline interpolation does not converge

$$\exists f \in C([0, 1]) \quad L_n f \not\rightarrow f \quad \text{as} \quad \max(t_{j+1} - t_j) \rightarrow 0.$$

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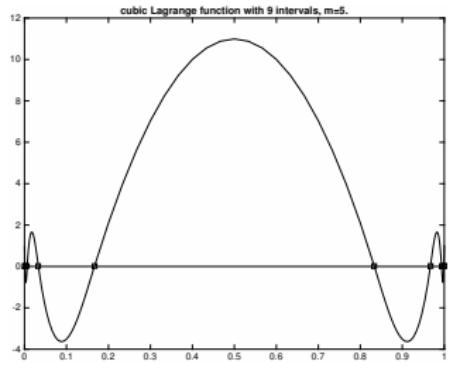
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- ▶ Set $t_{k+1} = t_k + M(t_k - t_{k-1})$
- ▶ Let $H_i := t_{i+1} - t_i$.
Then $H_k = M^k H_0$.
- ▶ Lagrange function at $t_0 = 0$.
- ▶ Growth restriction: $M = \max_{|i-j| \leq 1} \frac{t_{i+1} - t_i}{t_{j+1} - t_j}$
 $\iff H_j \leq C_M H_i \left(1 + \frac{|t_i - t_j|}{H_i}\right)$.



Local density

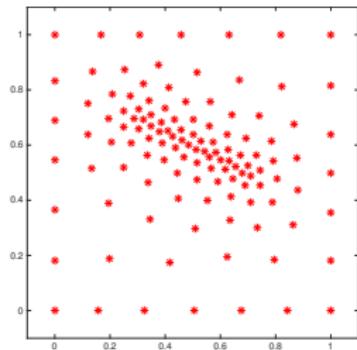
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$$H : \Omega \rightarrow (0, \infty)$$

admitting *local polynomial reproduction*

$$a : \Xi \times \Omega \rightarrow \mathbb{R} : (\xi, \alpha) \mapsto a(\xi, \alpha)$$

- ▶ $\forall p \in \Pi_\ell, \sum_{\xi \in \Xi} a(\xi, \alpha)p(\xi) = p(\alpha)$
- ▶ $\forall \alpha, \text{supp } a(\cdot, \alpha) \subset B(\alpha, H(\alpha))$
- ▶ $\forall \alpha, \sum_{\xi \in \Xi} |a(\xi, \alpha)| \leq K$



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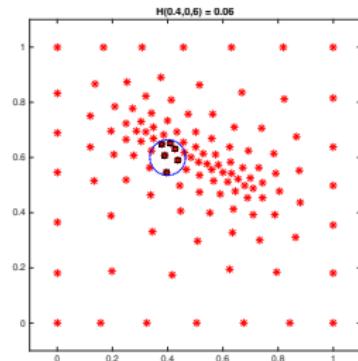
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Local density at $\alpha = (.4, .6)$
admitting LPR with $\ell = 2$.

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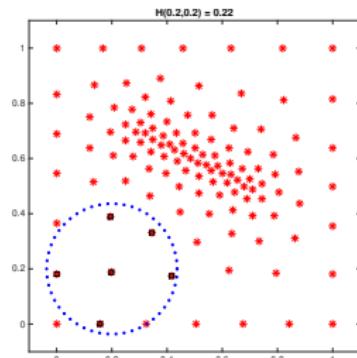
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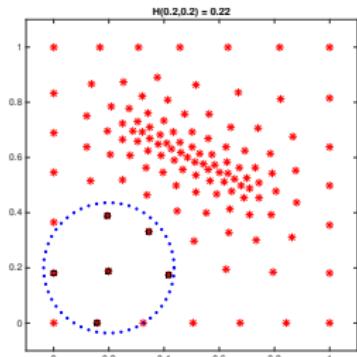
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[Wu-Schaback, '93] Suppose $\phi \in C^s(\mathbb{R}^d)$ is positive definite and $\ell \geq s$. For $f \in \mathcal{N}_\phi$, the RBF interpolant $I_\Xi f \in \text{span}_{\xi \in \Xi} \phi(\cdot - \xi)$ satisfies

$$|f(x) - I_\Xi f(x)| \leq C(H(x))^{s/2} \|f\|_{\mathcal{N}_\phi}$$

Surface spline approximation

Surface splines: $\phi(x) := \begin{cases} |x|^{2m-d} \log |x|, & d \text{ even} \\ |x|^{2m-d}, & d \text{ odd.} \end{cases} \in C^{2m-d}(\mathbb{R}^d)$

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Fundamental solution: for $f \in C_C^\infty(\mathbb{R}^d)$, $f(x) = \Delta^m f * \phi(x)$

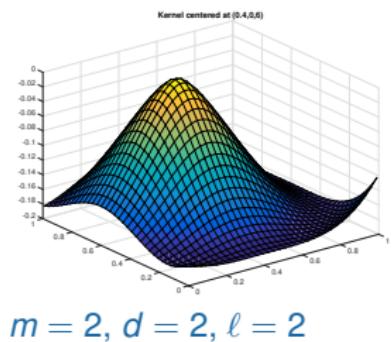
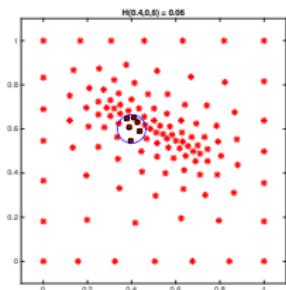
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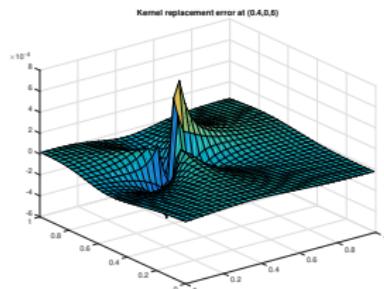
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Localized by LPR: If $a : \Xi \times \mathbb{R}^d \rightarrow \mathbb{R}$ reproduces Π_ℓ , then

$$\left| \phi(x - \alpha) - \sum_{\xi \in \Xi} a(\xi, \alpha) \phi(x - \xi) \right| \leq C(H(\alpha))^{2m-d} \left(1 + \frac{|x - \alpha|}{H(\alpha)} \right)^{2m-d-\ell-1}$$



$$m = 2, d = 2, \ell = 2$$



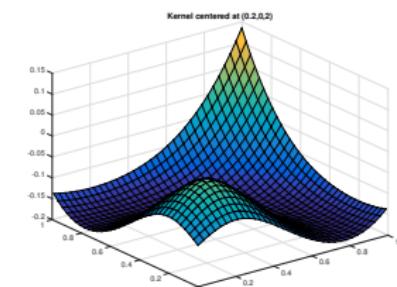
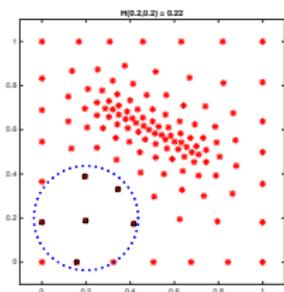
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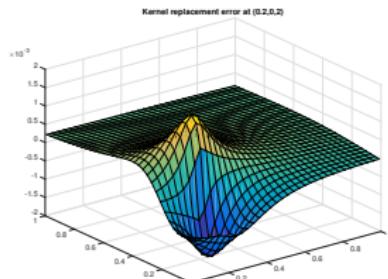
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DeVore-Ron '10: Replace $\phi(x - \alpha)$ by $\sum_{\xi \in \Xi} a(\xi, \alpha) \phi(x - \xi)$

$$f = \Delta^m f * \phi \rightsquigarrow T_{\Xi} f(x) = \int_{\mathbb{R}^d} \Delta^m f(\alpha) \left(\sum_{\xi \in \Xi} a(\alpha, \xi) \phi(x - \xi) \right) d\alpha$$

Error in terms of *majorant*: Suppose $H : \mathbb{R}^d \rightarrow (0, \infty)$ is a local density function admitting a LPR $a : \Xi \times \mathbb{R}^d \rightarrow \mathbb{R}$ of order ℓ , and

$$\tilde{H}(x) := \sup_{\alpha \in \mathbb{R}^d} H(\alpha) \left(1 + \frac{|x - \alpha|}{H(\alpha)} \right)^{-r}, \quad 0 < r < \frac{\ell - 2m}{2m - d}$$

then

$$|f(x) - T_{\Xi} f(x)| \lesssim (\tilde{H}(x))^{\sigma} \|f\|_{C^{\sigma}(\mathbb{R}^d)}, \quad 0 < \sigma \leq 2m$$

Local densities with slow growth

A local density $H : \mathbb{R}^d \rightarrow \mathbb{R}_+$ has slow growth $0 < \epsilon \leq 1$ if

$$\forall x, \alpha \quad H(\alpha) \leq CH(x) \left(1 + \frac{|x - \alpha|}{H(x)}\right)^{1-\epsilon}$$

- For knot sequences in \mathbb{R} , $H(i) = t_{i+1} - t_i$, $M = \max_{|i-j| \leq 1} \frac{h(j)}{h(j)}$,

$$m < \infty \quad \text{iff} \quad H(j) \leq CH(i) \left(1 + \frac{|t_i - t_j|}{H(i)}\right).$$

By Marsden's example, this is not always enough.

- The majorant of DeVore-Ron satisfies this. In fact

$$H(\alpha) \left(1 + \frac{|x - \alpha|}{H(\alpha)}\right)^{-r} \lesssim H(x) \quad \text{iff} \quad H(\alpha) \lesssim H(x) \left(1 + \frac{|x - \alpha|}{H(x)}\right)^{1-\epsilon}$$

with $\epsilon = \frac{1}{r+1}$

Main Result

Suppose $\Omega \subsetneq \Omega_L$, $\Xi_L \subset \Omega_L$, $\Xi := \Xi_L \cap \Omega$.

For $0 < \epsilon \leq 1$ there is $\nu > 0$ so that if

- ▶ $H : \Omega_L \rightarrow (0, \infty)$ has ϵ slow growth,
- ▶ is a local density admitting a LPR of degree $\ell > (2m - d)/\epsilon$,
- ▶ satisfies $\frac{H(\xi)}{q(\xi)} \leq M$ where $q(\xi) := \min_{\zeta \in \Xi \setminus \{\xi\}} \text{dist}(\xi, \zeta)$,

then for $\xi \in \Xi$, $\chi_\xi \in S(\phi, \Xi_L)$

$$|\chi_\xi(x)| \lesssim M^{m-d/2} \exp \left[-\nu \left(\frac{\text{dist}(\xi, x)}{H(\xi)} \right)^\epsilon \right], \quad \text{for } x \in \Omega.$$

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The **penalized Lebesgue constant** $\mathcal{L}_{\Xi, \sigma} := \sup_{x \in \Omega} \sum_{\xi \in \Xi} |\chi_\xi(x)| \left(1 + \frac{|x-\xi|}{H(x)}\right)^\sigma$ is bounded (in terms of $M, m, d, \epsilon, \sigma$).

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The penalized Lebesgue constant controls the operator norm of I_Ξ :

$$\|I_\Xi f/H^\sigma\|_\infty \leq \mathcal{L}_{\Xi, \sigma} \|f/H^\sigma\|_\infty$$

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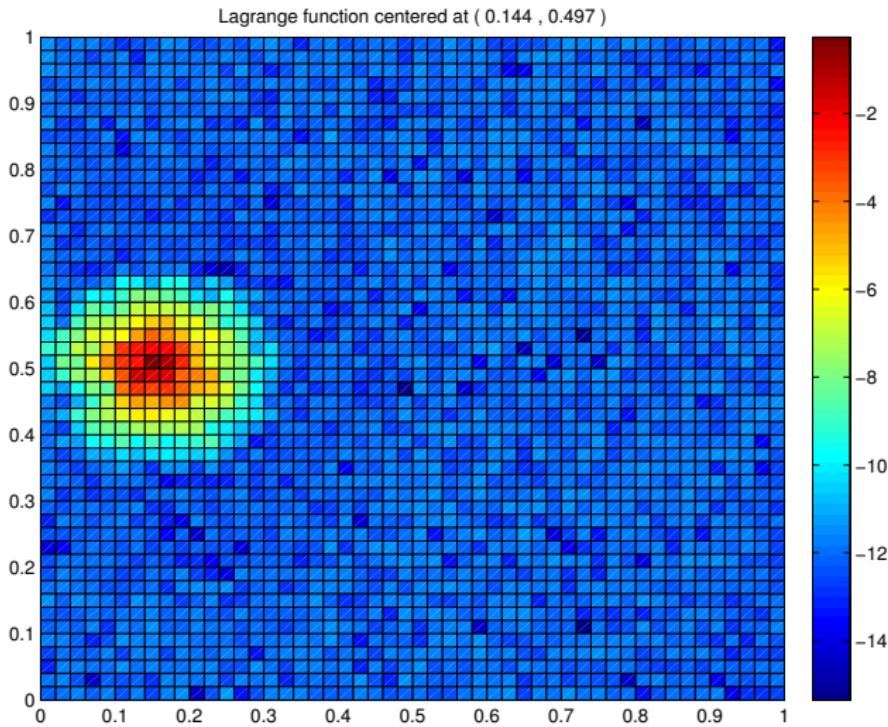
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$$\text{supp}(f) \subset \Omega \implies |f(x) - I_\Xi f(x)| \lesssim (H(x))^\sigma \|f\|_{C^\sigma(\mathbb{R}^d)}, \quad 0 < \sigma \leq 2m.$$

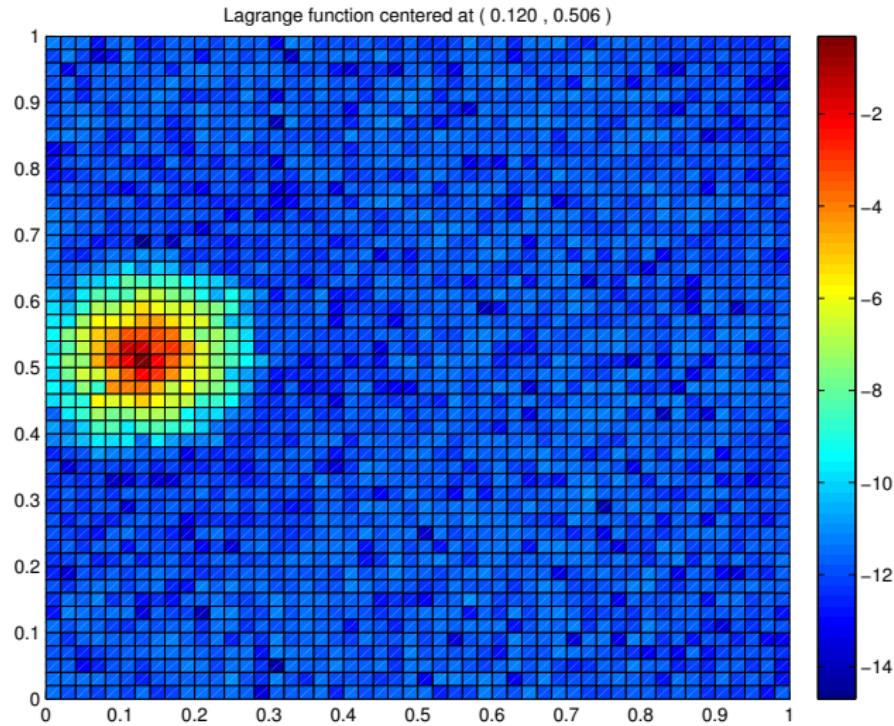
Boundary effects for Lagrange functions

A Lagrange function centered in the interior of $[0, 1]^2$.



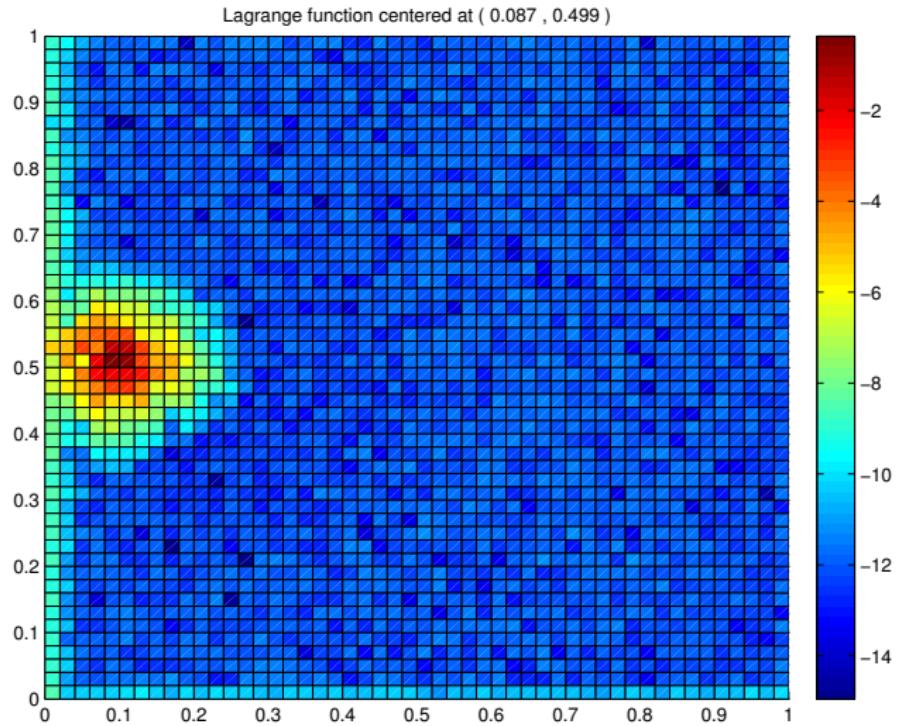
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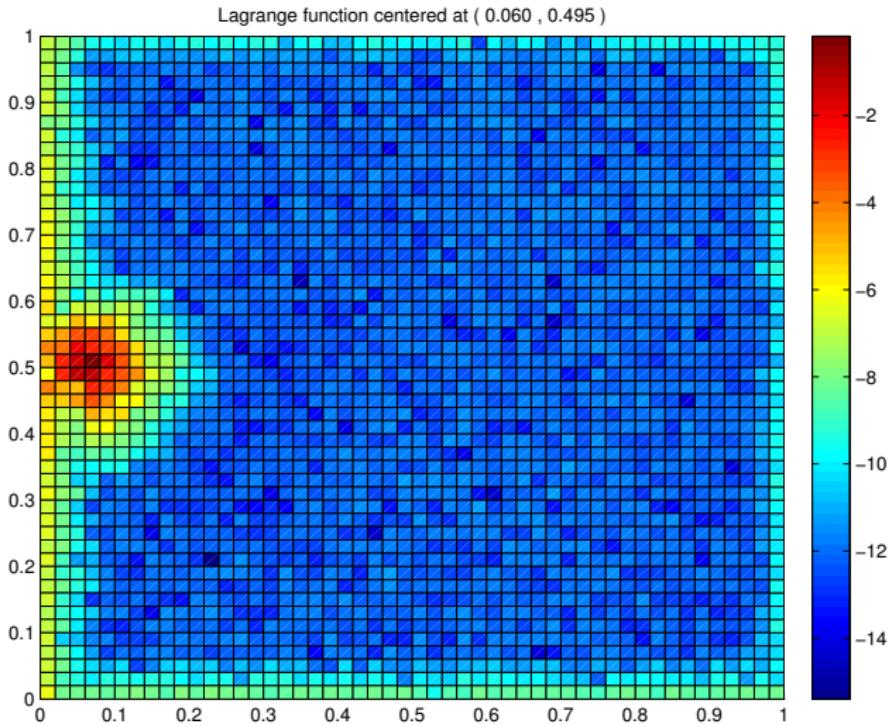
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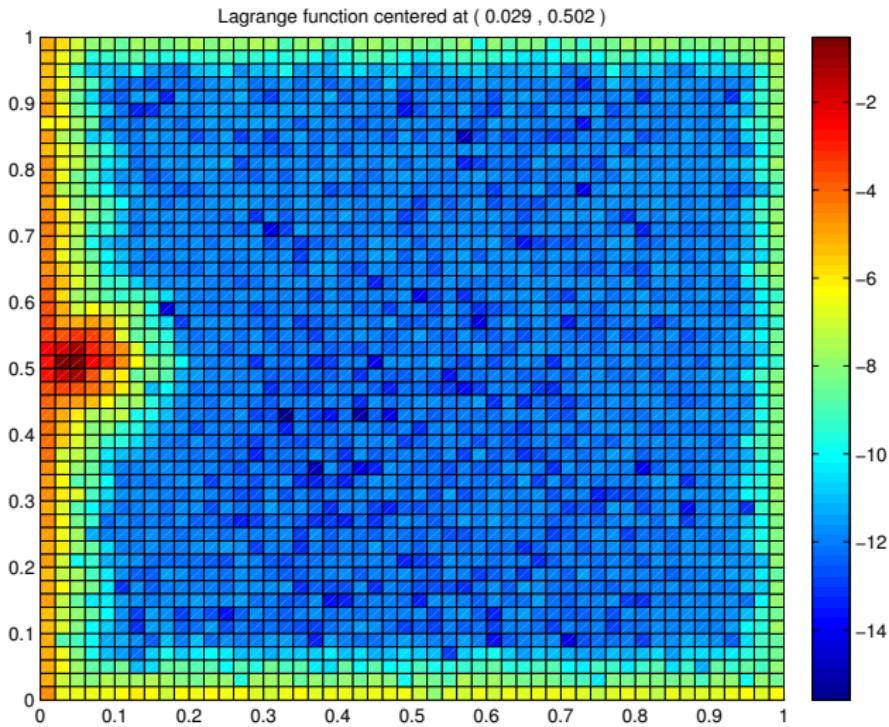
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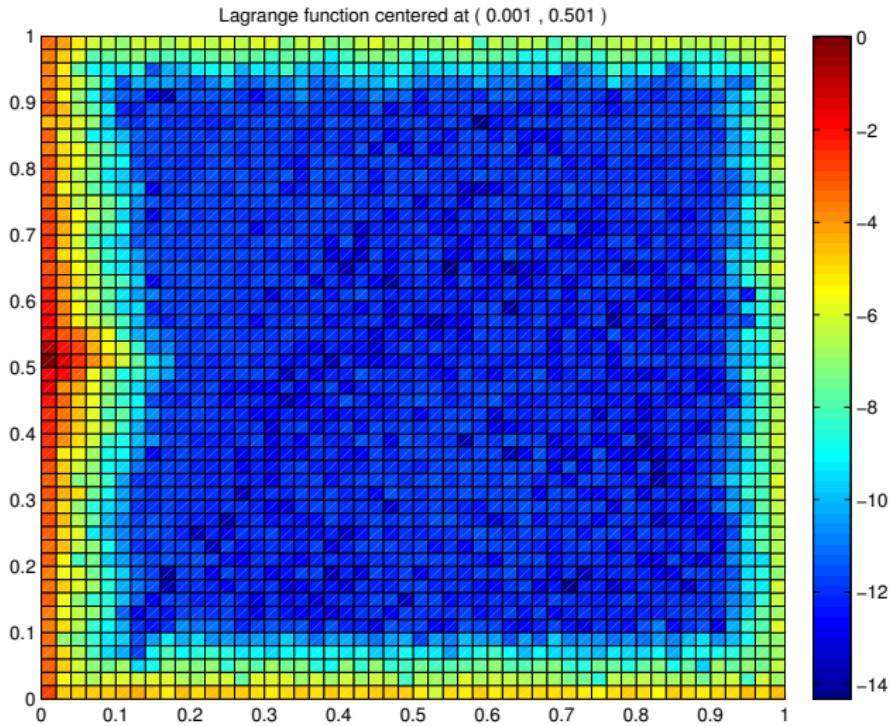
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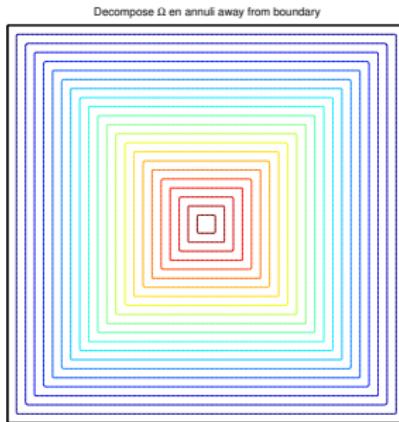
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How important is it to be “boundary-free”?

For compact $\Omega \subset \mathbb{R}^d$, and $0 \leq R$, $\Omega_R := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \geq R\}$.

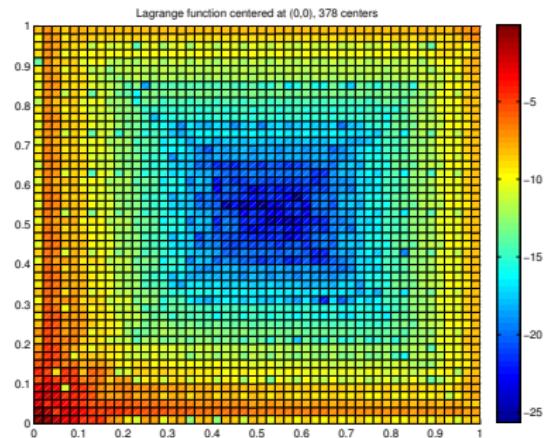
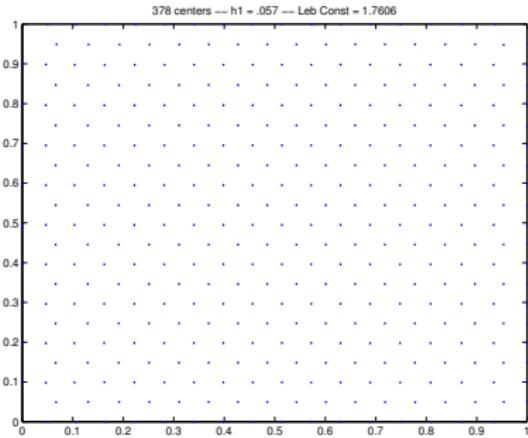


There exist positive constants C, h_0 and ν depending only on $\partial\Omega$ and m so that for $h < h_0$ we have

$$|\chi_\xi(x)| \leq Ce^{-\nu \frac{\max(\text{dist}(x, \partial\Omega) - \text{dist}(\xi, \partial\Omega), 0)}{h}}.$$

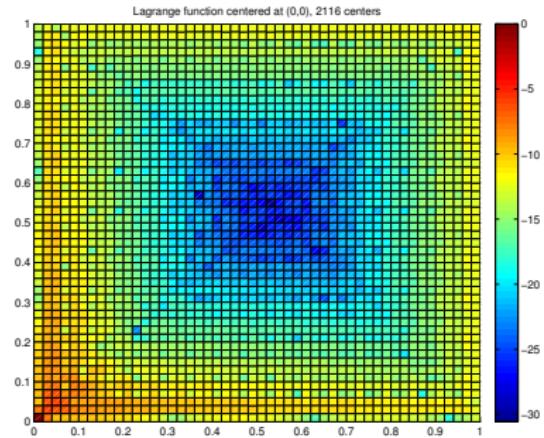
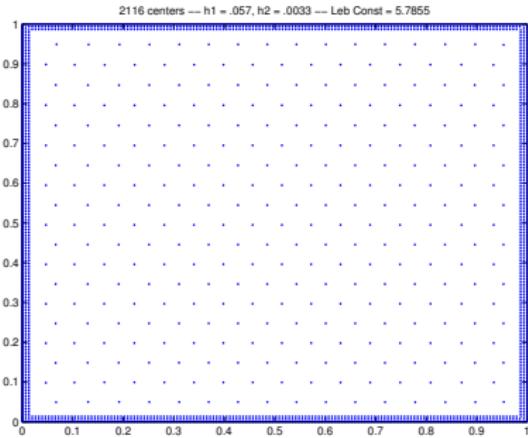
Question: For $\xi \in \partial\Omega$, does $|\chi_\xi(x)|$ decay along boundary?

A final experiment



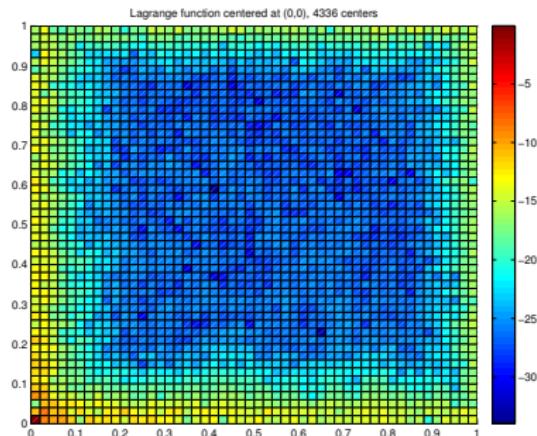
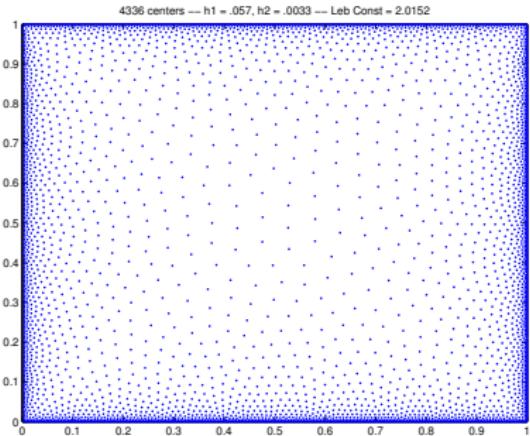
378 relatively equispaced centers, and the (log-scaled)
Lagrange function centered at the origin.

A final experiment



Adding three rings of centers near the boundary, with spacing $h_2 = .0033$.

A final experiment



Replacing these with nicely varying centers (created with DistMesh), having $.0033 \leq h(x) \leq .057$, and satisfying

$$h(x) \lesssim h(y)(1 + \frac{|x - y|}{h(y)})^{7/12}$$

(i.e., $\epsilon = 5/12$).