# On boundaries in approximation by polyharmonic kernels

Thomas Hangelbroek

University of Hawaii at Manoa

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# Error estimates for kernel approximation

Given a metric space $\Omega$ and a kernel $k : \Omega \times \Omega \to \mathbb{R}$

- $k$ is positive definite if for any finite set of centers $\Xi$, the collocation matrix

$$\mathsf{C}_\Xi := \big(k(\xi,\zeta)\big)_{(\xi,\zeta)\in\Xi\times\Xi}$$

  is symmetric, positive definite.
- Interpolation: For $f \in C(\Omega)$,

$$I_\Xi f = \sum_{\xi\in\Xi} c_\xi k(\cdot,\xi)$$

  is the unique function in $\mathrm{span}_{\xi\in\Xi} k(\cdot,\xi)$ so that $I_\Xi f\,|_\Xi = f\,|_\Xi$.
- Native space: There is a Hilbert space of continuous functions $\mathcal{N}$ with $k$ as its reproducing kernel: $f(x) = \langle f, k(x,\cdot)\rangle_{\mathcal{N}}$
- Error estimate:

$$\|f - I_\Xi f\|_{\mathcal{N}}^2 + \|I_\Xi f\|_{\mathcal{N}}^2 = \|f\|_{\mathcal{N}}^2$$

# Error for kernels with $\mathcal{N} \subset W_2^m(\mathbb{R}^d)$

Let $\Omega$ be compact with smooth boundary.
Let $h := \max_{x\in\Omega} \operatorname{dist}(x, \Xi)$.

If $\mathcal{N} \subset W_2^m(\mathbb{R}^d)$ and $m > d/2$ then

$$\|f - I_\Xi f\|_{L_\infty(\Omega)} \lesssim h^{m-d/2} \|f - I_\Xi f\|_{W_2^m(\Omega)}$$

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In "boundary free" settings the rates can increase:

- compactly supported $f$ (Bejancu)
- 'doubling trick' (Schaback)
- shift-invariant approximation over $\Omega = \mathbb{R}^d$ (Buhmann, others)

# Surface Spline Approximation in $\mathbb{R}^d$

For compact $\Omega \subset \mathbb{R}^d$ and $\Xi \subset \Omega$

$$S(\phi_m, \Xi) := \left\{ \sum_{\xi \in \Xi} A_\xi \phi_m(\cdot - \xi) \,\middle|\, \forall p \in \mathcal{P}_{m-1} \sum_{\xi \in \Xi} A_\xi p(\xi) = 0 \right\} + \mathcal{P}_{m-1}$$

the surface splines:

$$\phi_m(x-\alpha) := \begin{cases} |x-\alpha|^{2m-d} \log|x-\alpha|, & d \text{ even} \\ |x-\alpha|^{2m-d}, & d \text{ odd.} \end{cases}$$

- Native space approximation order: For $f \in W_2^m(\Omega)$,
  $\mathrm{dist}(f, S(\phi_m, \Xi))_{L_p(\Omega)} = \mathcal{O}(h^{m-d(\frac{1}{2}-\frac{1}{p})_+})$
- Boundary free approximation order: If $f$ is supported away from $\partial\Omega$ – or if $\Xi$ is taken from a neighborhood of $\Omega$ – then for $f \in W_p^{2m}(\Omega)$, $\mathrm{dist}(f, S(\phi_m, \Xi))_{L_p(\Omega)} = \mathcal{O}(h^{2m})$ .

# Boundary effects

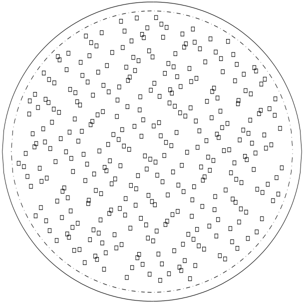
Let $\alpha > 0$. Consider $\Xi \subset \Omega$ satisfying

$$\operatorname{dist}(\Xi, \partial\Omega) > \alpha h \tag{1}$$

## Theorem (Johnson (98))

*For $\Xi$ satisfying (1) there is $f \in C^\infty(\overline{\Omega})$ so that*

$$\operatorname{dist}(f, S(\phi_m, \Xi))_p \neq o(h^{m+1/p}).$$



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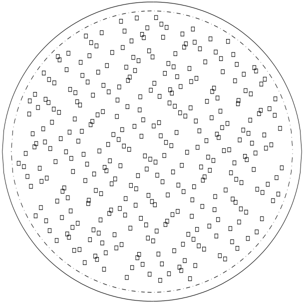
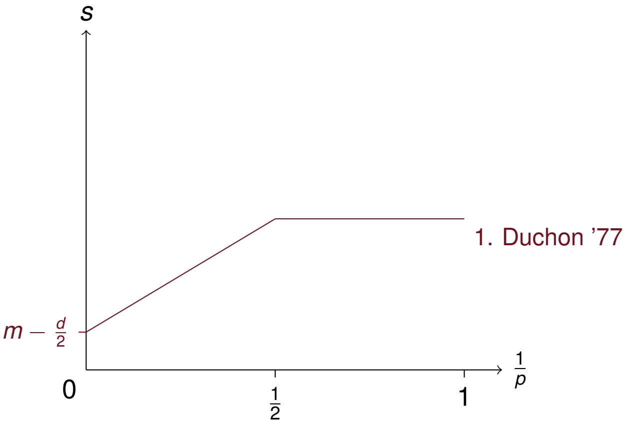


Q: Is $\mathcal{O}(h^{m+1/p})$ attainable for $f \in C^\infty(\overline{\Omega})$?

A: Yes, for $1 \leq p \leq 2$, and for $f \in B^{m+1/p}_{2,1}(\Omega)$ (Johnson '04)

Q: By violating (1) can we get $\mathcal{O}(h^{2m})$?

# Surface Spline Approximation Orders



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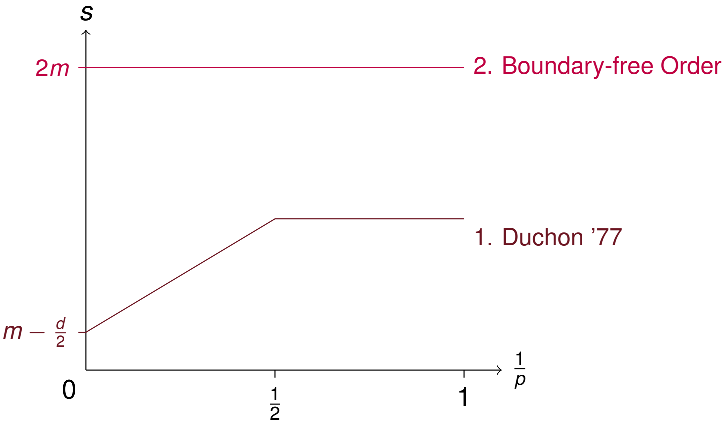

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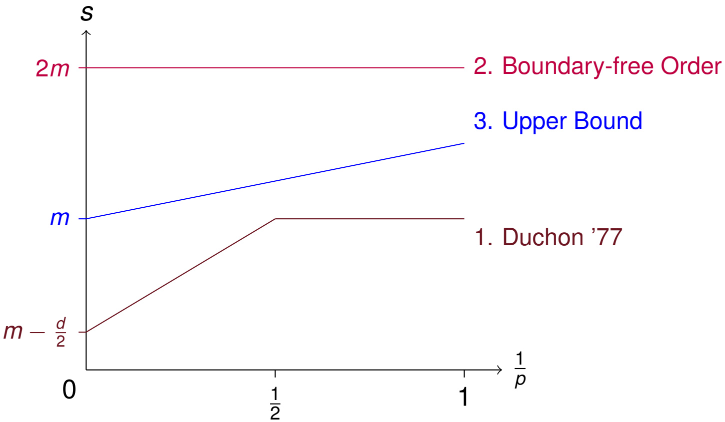

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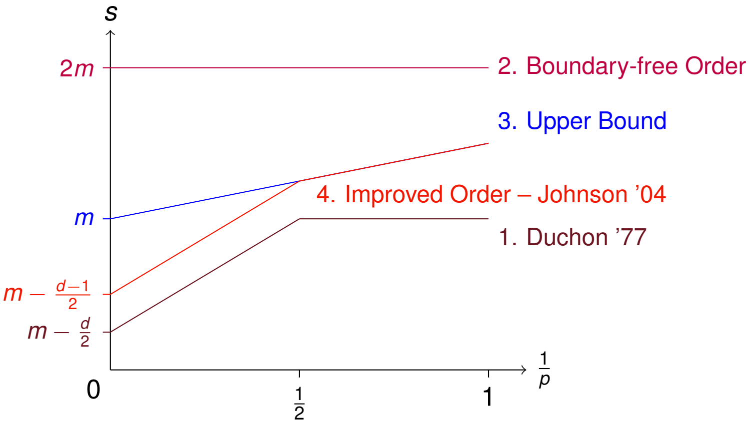


Q1: Is $m + 1/p$ the 'saturation order'? I.e. is $\mathcal{O}(h^{m+1/p})$ attainable?

Q2: Can we get $\mathcal{O}(h^{2m})$ by placing extra centers near boundary?

# Results

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary.

- For $1 < p < \infty$, and $f \in B^s_{p,1}(\Omega)$, $0 < s \leq m + 1/p$

$$\mathrm{dist}(f, S(\phi_m, \Xi))_p \lesssim h^s \|f\|_{B^s_{p,1}(\Omega)}.$$

- For $p = 1, \infty$ use $B^{s+\epsilon}_{1,\infty}(\Omega)$ or $C^{s+\epsilon}(\overline{\Omega})$

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- For $p = 1, \infty$ use $B^{s+\epsilon}_{1,\infty}(\Omega)$ or $C^{s+\epsilon}(\overline{\Omega})$
- Use two fill distances:
  - $h_1 = h(\Omega, \Xi)$ – the global fill distance.
  - $h_2$ local fill distance around $\partial\Omega$. (In a $Kh_2$ neighborhood of $\partial\Omega$.)

  Then, for $f \in W^{2m}_p(\Omega)$ (or $C^{2m}(\overline{\Omega})$ when $p = \infty$)

$$\operatorname{dist}(f, S(\phi_m, \Xi))_p \lesssim \; (h_1^{2m} + h_2^{m+\frac{1}{p}}) \|f\|_{W^{2m}_p(\Omega)}.$$

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- For $p = \infty$, if $h_2 \le h_1^2$,

$$\mathrm{dist}(f, S(\phi_m, \Xi))_\infty \lesssim h_1^{2m}\|f\|_{C^{2m}(\overline{\Omega})}$$

# Integral Representation for Bounded Domains

$$f(x) = \int_\Omega \mathcal{L}_m f(\alpha) k(x,\alpha)\,\mathrm{d}\alpha + \sum_{j=0}^{m-1} \int_{\partial\Omega} N_j f(\alpha)\ \ \lambda_{j,\alpha} k(x,\alpha)\,\mathrm{d}\sigma(\alpha) + p(x)$$

$\mathcal{L}_m = \Delta^m$ or $(1-\Delta)^m$;
$k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ – its fundamental solution.
$\Omega \subset \mathbb{R}^d$ compact with smooth boundary.
$f \in C^{2m}(\overline{\Omega})$ and $x \in \Omega$

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$$=$$

$$\int_{\Omega} \mathcal{L}_m f(\alpha) k(x,\alpha)\,\mathrm{d}\alpha + \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[ S_j f(\alpha) \lambda_j k(x,\alpha) - \lambda_j f(\alpha) S_j k(x,\alpha) \right] \mathrm{d}\sigma(\alpha)$$

(Green's Representation)

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Dirichlet boundary operators $\lambda_j$, $j = 0 \ldots m-1$ :

$$D_n \Delta^{\frac{j-1}{2}}, \text{ or } \quad \mathrm{Tr}\Delta^{\frac{j}{2}}.$$

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Operator of "Order" $2m - j - 1$

$$N_j = \sum \psi \mathrm{Tr} B,$$

- $B$ a differential operator,
- $\mathrm{Tr}$ trace on the boundary,
- $\psi$ a pseudodifferential operator on boundary.
- $\mathrm{Order}(B) + \mathrm{Order}(\psi) \leq 2m - j - 1.$

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$$K(x,\alpha) \qquad\qquad\qquad K_j(x,\alpha)$$

Approximation scheme:

Replace $k(x,\alpha)$ by $K(x,\alpha) = \sum_{\xi\in\Xi} c(\alpha,\xi)k(x,\xi)$

Replace each $\lambda_{j,\alpha}k(x,\alpha)$ by $K_j(x,\alpha) = \sum_{\xi\in\Xi} c_j(\alpha,\xi)k(x-\xi)$

$$s_f(x) := \sum_{\xi\in\Xi} A_\xi k(x,\xi) + p$$

with

$$A_\xi = C_{m,d} \int_\Omega c(\alpha,\xi)\Delta^m f(\alpha)\,\mathrm{d}\alpha + \sum_{j=0}^{m-1} \int_{\partial\Omega} c_j(\alpha,\xi) N_j f(\alpha)\mathrm{d}\sigma(\alpha)$$

# Dirichlet Problem via Boundary Layer Potentials

Find a solution of the Dirichlet Problem

$$\begin{cases} \mathcal{L}_m u(\alpha) = 0, & \alpha \in \Omega; \\ \lambda_j u(\alpha) = h_j(\alpha) & \alpha \in \partial\Omega, j = 0, \ldots, m-1; \end{cases}$$

using boundary layer potentials $V_j g(x) := \int_{\partial\Omega} \lambda_{j,\alpha} k(x,\alpha) g(\alpha) \mathrm{d}\alpha$. I.e., of the form

$$u(x) = \sum_{j=0}^{m-1} V_j g_j(x) = \sum_{j=0}^{m-1} \int_{\partial\Omega} \lambda_{j,\alpha} k(x,\alpha) g_j(\alpha) \, \mathrm{d}\sigma(\alpha).$$

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Equivalently: solve the system of integral equations:

$$\widetilde{L} \begin{pmatrix} g_0 \\ \vdots \\ g_{m-1} \end{pmatrix} := \begin{pmatrix} \lambda_0 \big( V_0 g_0 + V_1 g_1 + \cdots + V_{m-1} g_{m-1} \big) \\ \vdots \\ \lambda_{m-1} \big( V_0 g_0 + V_1 g_1 + \cdots + V_{m-1} g_{m-1} \big) \end{pmatrix} = \begin{pmatrix} h_0 \\ \vdots \\ h_{m-1} \end{pmatrix}$$

# System of integral equations

$\widetilde{L} : (\mathcal{D}'(\partial\Omega))^m \to (\mathcal{D}'(\partial\Omega))^m$ is a pseudodifferential operator. It is elliptic and that the augmented operator

$$L := \left(\begin{array}{c|c} \widetilde{L} & P \\ \hline P^* & 0 \end{array}\right) \quad \text{where} \quad P = \begin{pmatrix} \lambda_0 p_1 & \dots & \lambda_0 p_N \\ \vdots & \ddots & \vdots \\ \lambda_{m-1} p_1 & \dots & \lambda_{m-1} p_N \end{pmatrix},$$

($p_1 \dots p_N$ a basis for $\mathcal{P}_{m-1}$) is boundedly invertible from

$$\mathcal{A}_{p,s} := W_p^s(\partial\Omega) \times \dots \times W_p^{s+m-1}(\partial\Omega) \times \mathbb{R}^N$$

to

$$\mathcal{B}_{p,s+2m-1} := W_p^{s+2m-1}(\partial\Omega) \times \dots \times W_p^{s+m}(\partial\Omega) \times \mathbb{R}^N$$

for any $s \in \mathbb{R}$, $1 < p < \infty$. The solution $\mathbf{g} = (g_0 \dots g_{m-1})^T$ and the coefficients $\mathbf{a} = (a_1 \dots a_N)^T$ of $p = \sum a_j p_j$ are

$$\begin{pmatrix} \mathbf{g} \\ \mathbf{a} \end{pmatrix} = L^{-1} \begin{pmatrix} \mathbf{h} \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{h} = (\lambda_0 f \dots \lambda_{m-1} f)^T$$

1. For any $s \in \mathbb{R}$ and $1 < p < \infty$, $\widetilde{L}$ is bounded from $W_p^s(\partial\Omega) \times \cdots \times W_p^{s+m-1}(\partial\Omega)$ to $W_p^{s+2m-1}(\partial\Omega) \times \cdots \times W_p^{s+m}(\partial\Omega)$
2. It is self-adjoint

$$
\begin{aligned}
(\widetilde{L})^* \quad : \quad & \left(W_p^{s+2m-1} \times \cdots \times W_p^{s+m}\right)' \longrightarrow \left(W_p^{s} \times \cdots \times W_p^{s+m-1}\right)' \\
& \left(W_{p'}^{-s+1-2m} \times \cdots \times W_{p'}^{-s-m}\right) \to \left(W_{p'}^{-s} \times \cdots \times W_{p'}^{-s-m+1}\right)
\end{aligned}
$$

3. The range of $\widetilde{L}$ is closed in $W_p^{s+2m-1} \times \cdots \times W_p^{s+m}$ (it has a right parametrix $\widetilde{L}R = I + K$).
4. Injectivity does not necessarily hold for $\widetilde{L}$, but it does for
$$
L = \left(\begin{array}{c|c} \widetilde{L} & P \\ \hline P^* & 0 \end{array}\right)
$$
5. $\mathrm{ran} L = \overline{\mathrm{ran} L} =^{\perp} \ker L^* =^{\perp} \{0\}$

END