

# Kernel approximation, elliptic PDE and boundary effects

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# Surface Spline Approximation in $\mathbb{R}^d$

Surface splines:

$$\phi_m(x) := \begin{cases} |x|^{2m-d} \log |x|, & d \text{ even} \\ |x|^{2m-d}, & d \text{ odd.} \end{cases}$$

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Fundamental solution: for  $f \in C^\infty(\mathbb{R}^d)$  with compact support,

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Approximation scheme:  $\phi_m \rightsquigarrow \tilde{k}(x, \alpha) = \sum_{\xi \in \Xi} a(\alpha, \xi) \phi_m(x - \xi)$

$$T_\Xi f(x) = C_{m,d} \int_{\mathbb{R}^d} \Delta^m f(\alpha) \left( \sum_{\xi \in \Xi} a(\alpha, \xi) \phi_m(x - \xi) \right) \, d\alpha$$

$$T_\Xi f = \sum_{\xi \in \Xi} A_\xi \phi_m(\cdot - \xi) \in \text{span}_{\xi \in \Xi} \phi_m(\cdot - \xi) =: S(\phi_m, \Xi)$$

## Replacing kernel

Replace kernel  $\phi_m(x - \alpha) \rightsquigarrow \tilde{k}(x, \alpha) = \sum_{\xi \in \Xi} a(\alpha, \xi) \phi_m(x - \xi)$  by using a “local polynomial reproduction” so that

$$(\forall p \in \mathcal{P}_L) \sum_{\xi \in \Xi} a(\alpha, \xi) p(\xi) = p(\alpha) \quad \text{and} \quad \text{supp} \left( \sum_{\xi \in \Xi} a(\alpha, \xi) \delta_\xi \right) \subset B(\alpha, H).$$

$H$  can be proportional to fill-distance  $h := \max_{x \in \text{supp}(f)} \min_{\xi \in \Xi} |x - \xi|$

[Wu-Schaback] and [Devore-Ron] permit  $H$  to depend on  $\alpha$

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- ▶ Error is small and bell shaped ([Dyn-Levin-Rippa], [Rabut])

$$Err(x, \alpha) = |\phi(x - \alpha) - \tilde{k}(x, \alpha)| \lesssim H^{2m-d} \left( 1 + \frac{|x - \alpha|}{H} \right)^{2m-d-L}$$

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$$|f(x) - T_{\Xi} f(x)| \leq \int_{\mathbb{R}^d} |\Delta^m f(\alpha)| Err(x, \alpha) d\alpha =: (E \Delta^m f)(x)$$

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Basic Error Estimate :  $\|f - T_{\Xi} f\|_p \leq \|E(\Delta^m f)\|_p \leq C H^{2m} \|\Delta^m f\|_p$

# General Strategy

- ▶ Represent functions with an integral using kernel
- ▶ Replace kernel in integral with combination of scattered shifts

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**Example:** On  $\mathbb{S}^d$ ,  $k(x, y) = \begin{cases} (1 - x \cdot y)^{m-d/2} & d \text{ odd} \\ (1 - x \cdot y)^{m-d/2} \log(1 - x \cdot y) & d \text{ even} \end{cases}$

$$f(x) = \int_{\mathbb{S}^d} \mathcal{L}f(y)k(x, y)dy + p(x), \quad \mathcal{L} = \prod_{j=1}^m (\Delta - r_j).$$

Replace  $k(x, y)$  by  $\sum a(y, \xi)k(x, \xi)$  using a local spherical harmonic reproduction  $\sum a(y, \xi)\delta_\xi$ .  $Err(x, y) \lesssim H^{2m-d} \left(1 + \frac{\text{dist}(x, y)}{H}\right)^{-d-1}$

$$T_\Xi f = \sum_{\xi \in \Xi} \left( \int_{\mathbb{S}^d} \mathcal{L}f(y)a(y, \xi)dy \right) k(\cdot, \xi) + p \text{ has approx. order } 2m$$

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Example [H-Schmid,'11]: On  $SO(3)$   $k(x, y) = \left(\sin\left(\frac{\omega(x, y)}{2}\right)\right)^{m-3/2}$

$$f(x) = \int_{SO(3)} \mathcal{L}f(y)k(x, y)dy + p(x), \quad \mathcal{L} = \prod_{j=1}^m (\Delta - r_j).$$

Replace  $k(x, y)$  by  $\sum a(y, \xi)k(x, \xi)$  using a local eigenfunction reproduction  $\sum a(y, \xi)\delta_\xi$ :  $Err(x, y) \lesssim H^{2m-d} \left(1 + \frac{\text{dist}(x, y)}{H}\right)^{-4}$

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# Kernel approximation on manifolds

$\mathbb{M}$  - compact,  $d$ -dimensional Riemannian manifold, without boundary.  
For  $\Omega \subset \mathbb{M}$ , the Sobolev space  $W_2^m(\Omega)$ :

$$\langle f, g \rangle_{W_2^m(\Omega)} := \sum_{k=0}^m \int_{\Omega} \langle \nabla^k f, \nabla^k g \rangle_x dx.$$

Locally metric-equivalent to Sobolev spaces on  $\mathbb{R}^d$  via exponential map:  $W_2^m(B) \sim W_2^m(\exp_x(B))$ :

$W_2^m(\Omega)$  is a reproducing kernel Hilbert space when  $m > d/2$ .

$$f(x) = \langle f, k(x, \cdot) \rangle_{W_2^m(\mathbb{M})} = \int_{\mathbb{M}} \mathcal{L}f(y)k(x, y)dy$$

where  $\mathcal{L} = \sum_{k=0}^m (\nabla^k)^* \nabla^k$ .

How to replace kernel?  $k(x, y) \rightsquigarrow \tilde{k}(x, y) = \sum_{\xi \in \Xi} a_\xi k(x, \xi)$

# Kernel approximation on manifolds

For  $\Xi \subset \mathbb{M}$ ,  $\xi \in \Xi$ , let  $\chi_\xi = \sum_{\zeta \in \Xi} A_{\xi, \zeta} k(x, \zeta)$ ,

$$\chi_\xi(\zeta) = \begin{cases} 1 & \zeta = \xi \\ 0 & \zeta \in \Xi \setminus \xi \end{cases}$$

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[H-Narcowich-Ward, '10] There is  $\nu > 0$  so that

- ▶  $|\chi_\xi(x)| \lesssim e^{-\nu(\frac{\text{dist}(x, \xi)}{h})}$ .
- ▶  $h := \max_{x \in \mathbb{M}} \min_{\zeta \in \Xi} |x - \zeta|$
- ▶  $|A_{\xi, \zeta}| = |\langle \chi_\xi, \chi_\zeta \rangle_{W_2^m(\mathbb{M})}| \leq q^{d-2m} e^{-\nu(\frac{\text{dist}(\xi, \zeta)}{h})}$ .
- ▶  $q = \min_{\zeta \neq \eta} \text{dist}(\eta, \zeta)$ .

To replace: use  $\Xi \cup \{y\}$  to construct  $\chi_y$ . Replace  $k(x, y)$  with

$$\tilde{k}(x, y) := - \sum_{\zeta \in \Xi \setminus \{y\}} \frac{A_{y, \zeta}}{A_{y, y}} k(x, \zeta). \text{ Then}$$

$$Err(x, y) = |k(x, y) + \sum_{\zeta \in \Xi \setminus \{y\}} \frac{A_{y, \zeta}}{A_{y, y}} k(x, \zeta)| = \left| \frac{\chi_y(x)}{A_{y, y}} \right| \lesssim h^{2m-d} e^{-\nu(\frac{\text{dist}(x, y)}{h})}$$

$$T_\Xi f = \sum_{\xi \in \Xi} \left( \int_{\mathbb{M}} \mathcal{L}f(y) \frac{A_{y, \xi}}{A_{y, y}} dy \right) k(\cdot, \xi) \text{ has approx. order } 2m$$

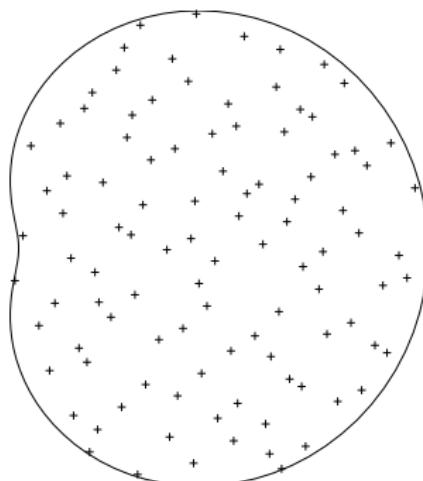
# Approximation on bounded regions

Consider  $\Xi \subset \Omega$ , where  $\Omega \subset \mathbb{R}^d$  is bounded, with smooth boundary.  
Approximate with surface splines

$$\phi_m(x) := \begin{cases} |x|^{2m-d} \log |x|, & d \text{ even} \\ |x|^{2m-d}, & d \text{ odd.} \end{cases}$$

## Theorem

[Johnson (98)] When centers cannot coalesce near  $\partial\Omega$ , then approximation order is saturated at  $m + 1/p$ .



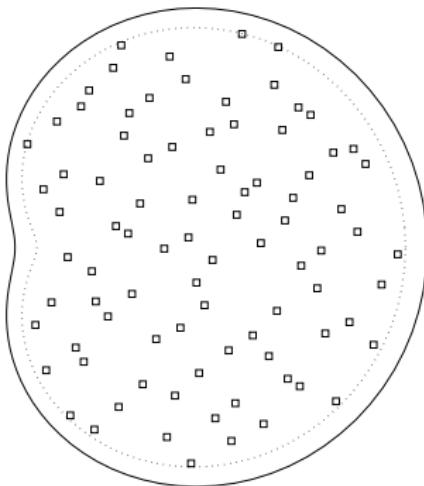
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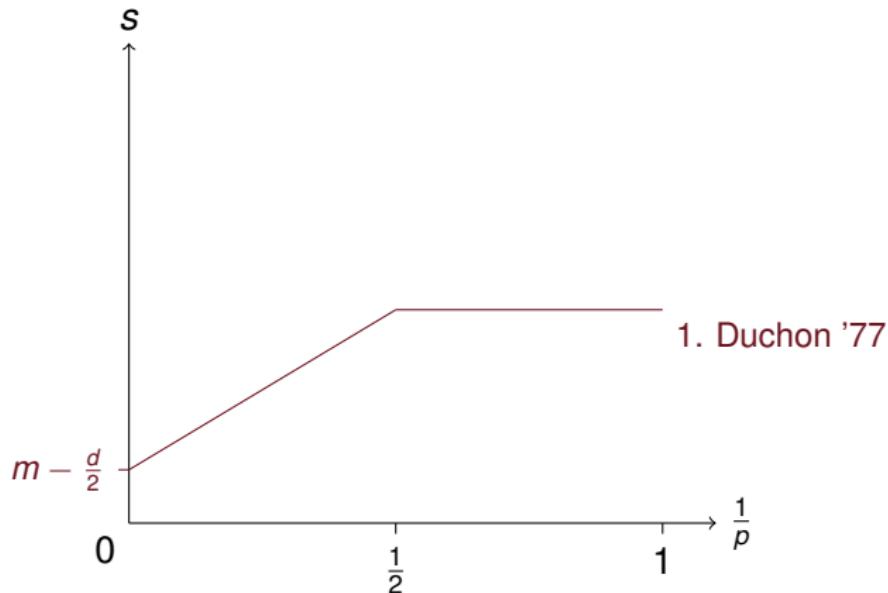
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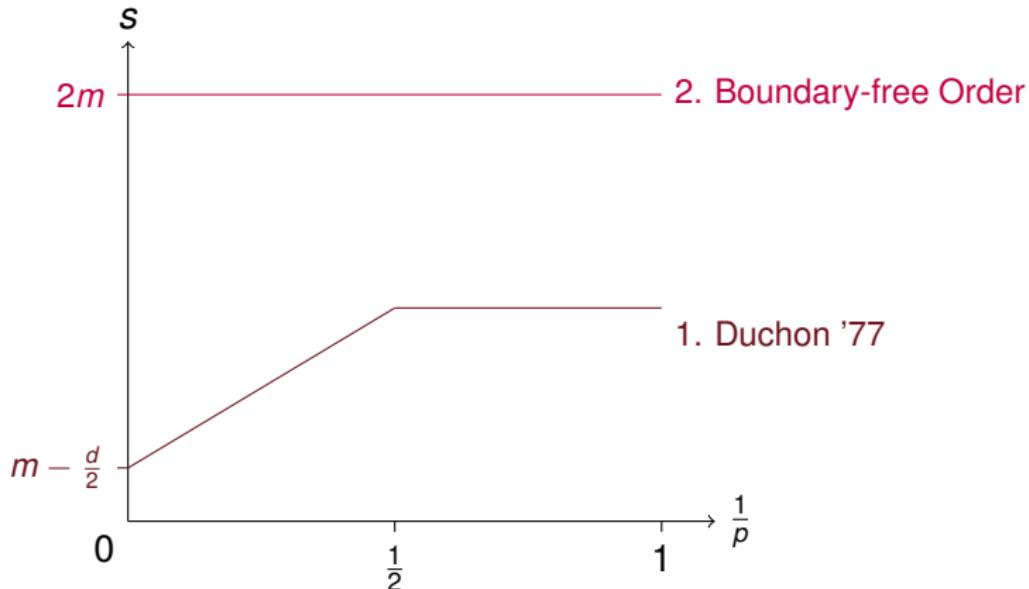
## Surface Spline Approximation Orders

1. For  $f \in W_2^m(\Omega)$ ,  $\text{dist}(f, S(\Xi))_p = \mathcal{O}(h^{m-(d/2-d/p)_+})$



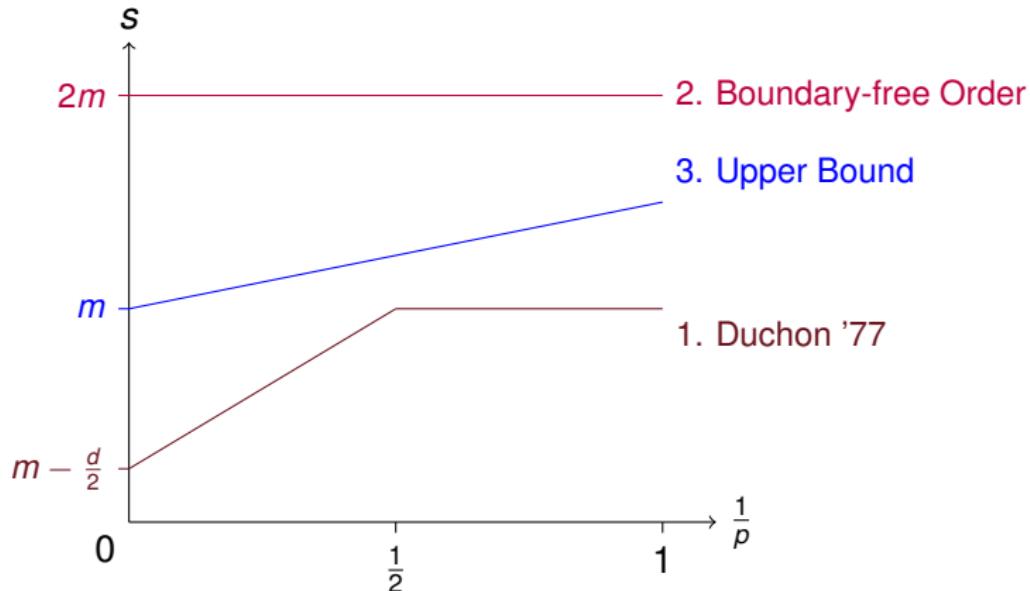
## Surface Spline Approximation Orders

2. For  $f \in W_p^{2m}(\mathbb{R}^d)$ ,  $\text{dist}(f, S(\Xi))_p = \mathcal{O}(h^{2m})$



## Surface Spline Approximation Orders

3. If  $\text{dist}(f, S(\Xi))_p = o(h^{m+1/p})$  then  $f$  is trivial.



# Integral Representation for Bounded Domains

Let  $\Omega \subset \mathbb{R}^d$  be compact with smooth boundary. For  $f \in C^{2m}(\overline{\Omega})$  and  $x \in \Omega$ :

$$\begin{aligned} f(x) &= \int_{\Omega} \Delta^m f(\alpha) \phi_m(x - \alpha) d\alpha \\ &+ \sum_{j=0}^{m-1} \int_{\partial\Omega} N_j f(\alpha) \lambda_{j,\alpha} \phi_m(x - \alpha) d\sigma(\alpha) + p(x) \end{aligned}$$

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λ<sub>j,α</sub> Operator of Order j

The operators  $\lambda_j, j = 0 \dots m - 1$  Dirichlet boundary operators:

$$D_n \Delta^{\frac{j-1}{2}}, \text{ or } \text{Tr} \Delta^{\frac{j}{2}}.$$

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Operator of "Order"  $2m - j - 1$

$$N_j = \sum \psi \text{Tr} B,$$

- ▶  $B$  a differential operator,
- ▶  $\text{Tr}$  trace on the boundary,
- ▶  $\psi$  a pseudodifferential operator on boundary.
- ▶  $\text{Order}(B) + \text{Order}(\psi) \leq 2m - j - 1$ .

# Polyharmonic Dirichlet Problem

Find a solution of the Dirichlet Problem

$$\begin{cases} \Delta^m u(\alpha) = 0, & \alpha \in \Omega; \\ \lambda_j u(\alpha) = \lambda_j f(\alpha) & \alpha \in \partial\Omega, j = 0, \dots, m-1; \end{cases}$$

using **boundary layer potentials**  $V_j g(x) := \int_{\partial\Omega} \lambda_{j,\alpha} \phi(x - \alpha) g(\alpha) d\alpha$ .  
I.e., of the form

$$u(x) = \sum_{j=0}^{m-1} V_j g_j(x) = \sum_{j=0}^{m-1} \int_{\partial\Omega} \lambda_{j,\alpha} \phi(x - \alpha) g_j(\alpha) d\sigma(\alpha).$$

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Equivalently: solve the system of integral equations:

$$\int_{\partial\Omega} K(x, \alpha) \begin{pmatrix} g_0(\alpha) \\ \vdots \\ g_{m-1}(\alpha) \end{pmatrix} d\alpha = \begin{pmatrix} \lambda_0 f(x) \\ \vdots \\ \lambda_{m-1} f(x) \end{pmatrix}$$

where  $K(x, \alpha) = (\lambda_{k,x} \lambda_{j,\alpha} \phi(x - \alpha))_{j,k}$ .

# Approximation scheme for Bounded Domains

$$T_{\Xi} f(x) = \int_{\Omega} \Delta^m f(\alpha) k(x, \alpha) d\alpha + \sum_{j=0}^{m-1} \int_{\partial\Omega} N_j f(\alpha) k_j(x, \alpha) d\sigma(\alpha) + p(x)$$

Replace  $\phi_m(x - \alpha)$  by  $\tilde{k}(x, \alpha) = \sum_{\xi \in \Xi} a(\alpha, \xi) \phi_m(x - \xi)$

Replace:  $\lambda_{j,\alpha} \phi_m(x - \alpha)$  by  $\tilde{k}_j(x, \alpha) = \sum_{\xi \in \Xi} a_j(\alpha, \xi) \phi_m(x - \xi)$

Error kernel:  $Err_j(x, \alpha) \lesssim H^{2m-d-j} \left(1 + \frac{|x-\alpha|}{H}\right)^{-d-1}$ .

Worst case:  $j = m-1$ .

$$\|E_j\|_{1 \rightarrow 1} = \max_{\alpha \in \partial\Omega} \int_{\Omega} Err_{m-1}(x, \alpha) dx \lesssim H^{2m-(m-1)} = H^{m+1}$$

$$\|E_j\|_{\infty \rightarrow \infty} = \max_{x \in \Omega} \int_{\partial\Omega} Err_{m-1}(x, \alpha) d\alpha \lesssim H^{2m-m} = H^m$$

Approximation orders:  $H \propto h$  throughout  $\Omega$ ,  $\|f - T_{\Xi} f\|_p = \mathcal{O}(h^{m+1/p})$ .  
If  $H \propto h^2$  near boundary then  $\|f - T_{\Xi} f\|_p = \mathcal{O}(h^{2m})$

END