Beyond Quasi-unifomity: Kernel Approximation with a Local Mesh Ratio

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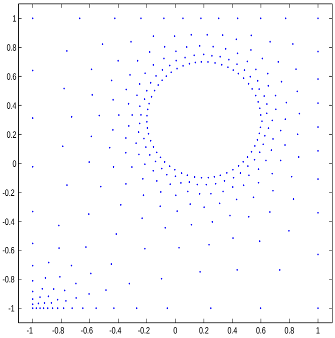
RBFs using centers with spatially varying density

Let $\Omega \subset \mathbb{R}^d$ and $\Xi \subset \Omega$ with $\#\Xi < \infty$. For a radial $k : \mathbb{R}^d \to \mathbb{R}$, approximate by using the finite dimensional spaces

$$S(\Xi) := \mathrm{span}_{\xi \in \Xi} k(\cdot - \xi).$$

- Results should not rely on $\rho = h/q$
- To treat realistic data, but also as a tool to treat boundaries
- How to measure $\mathrm{dist}(f, S(\Xi))_{L_p(\Omega)}$? Not $h := \max_{x \in \Omega} \mathrm{dist}(x, \Xi) \to 0$; error should depend on 'local density'
- Kernels are fundamental solutions of Δ^m and $(1 - \Delta)^m$ on $\mathbb{R}^d$.

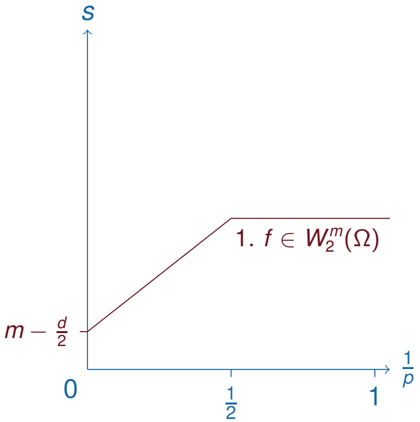


Boundary effects and error estimates

Ω: compact, smooth boundary. $h := \max_{x \in \Omega} \operatorname{dist}(x, \Xi)$.

1. For $f \in W_2^m(\Omega)$:

$$\operatorname{dist}(f, S(\Xi))_{L_p(\Omega)} = O(h^{m-(d/2-d/p)_+})$$



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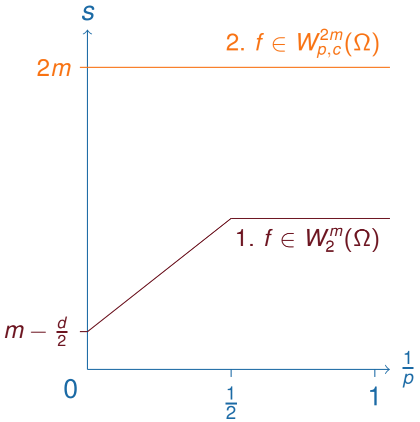
2. For $f \in W_{p,c}^{2m}(\Omega)$ (compact support in interior of Ω)

$$\operatorname{dist}(f, S(\Xi))_{L_p(\Omega)} = O(h^{2m})$$



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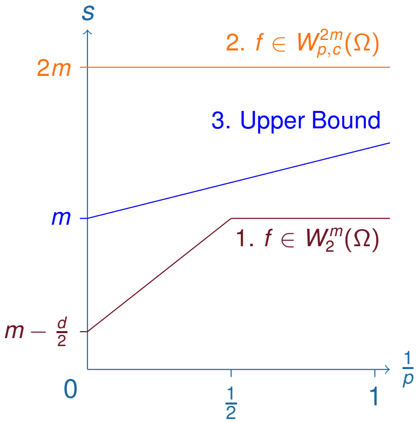
2. For $f \in W_{p,c}^{2m}(\Omega)$ (compact support in interior of Ω)

$$\mathrm{dist}(f, S(\Xi))_{L_p(\Omega)} = O(h^{2m})$$

3. For certain $\Xi \subset \Omega$ there are $f \in C^\infty(\overline{\Omega})$ so that

$$\mathrm{dist}(f, S(\Xi))_p \neq o(h^{m+1/p}).$$



Boundary effects

Theorem (Johnson (98))

Let $0 < \alpha < 1$. *For* $\Xi \subset \Omega$ *satisfying* $\mathrm{dist}(\Xi, \partial\Omega) > \alpha h$, *there is* $f \in C^\infty(\overline{\Omega})$ *so that*

$$\mathrm{dist}(f, S(\phi_m, \Xi))_p \neq o(h^{m+1/p}).$$

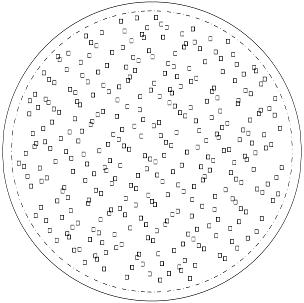
H('07)

Use Ξ with two fill distances:

- $h_1 = h(\Omega, \Xi)$ – the global fill distance.
- h_2 fill distance near $\partial\Omega$. (In a Kh_2 tube of $\partial\Omega$.)

For for $f \in C^{2m}(\overline{\Omega})$, if $h_2 \leq h_1^2$,

$$\mathrm{dist}(f, S(\Xi))_\infty \lesssim h_1^{2m} \|f\|_{C^{2m}(\overline{\Omega})}$$



Requires non-quasi-uniform points

RBF results with local parameter

A local density parameter is a function $h_m : \Omega \to [0, \infty)$ possessing a local polynomial reproduction of degree m:
a kernel $a : \Xi \times \mathbb{R}^d \to \mathbb{R} : (\xi, \alpha) \mapsto a(\xi, \alpha)$ satisfying

- For all $p \in \Pi_m$, $\sum_{\xi \in \Xi} a(\xi, \alpha) p(\xi) = p(\alpha)$.
- $a(\xi, \alpha) = 0$ when $|\xi - \alpha| > h_m(\alpha)$,
- $\sum_{\xi \in \Xi} |a(\xi, \alpha)| < K$ for all α.

Wu and Schaback ('93):
Given a local density parameter h_m, the error from interpolation satisfies

$$|f(x) - I_\Xi f(x)| \leq C\big(h_m(x)\big)^{m-d/2} \|f\|_{W_2^m(\mathbb{R}^d)}$$

RBF results with local parameter

DeVore and Ron ('10):
Suppose Ξ has local density parameter h_m possessing ϵ slow growth

$$\forall x, \alpha \in \mathbb{R}^d \qquad h_m(\alpha) \le C h_m(x) \left(1 + \frac{|x-\alpha|}{h_m(x)}\right)^{1-\epsilon}.$$

If $\sigma \le 2m$ and $f \in C_0^\sigma(\mathbb{R}^d)$, there is $s_f \in S(\Xi)$

$$|f(x) - s_f(x)| \lesssim \big(h_m(x)\big)^\sigma \|f\|_{C^\sigma}$$

In other words,

$$\mathrm{dist}_{L_\infty(h_m^{-\sigma})}\big(f, S(\Xi)\big) \lesssim \|f\|_{C^\sigma}$$

Where $L_\infty(h_m^{-\sigma})$ is the normed linear space with weighted norm $\|f\|_{L_\infty(h_m^{-\sigma})} = \|\frac{f}{h_m^\sigma}\|_\infty$.

No such results are known for bounded regions.

RBF results with local parameter

H('12):
If Ξ has local density parameter h_m possessing ϵ slow growth and local quasi-uniformity:

$$\forall \xi \in \Xi \qquad \frac{h_m(\xi)}{\min_{\zeta \neq \xi}(\xi, \zeta)} \leq \rho$$

then

$$|\chi_\xi(x)| \lesssim \rho^{m-d/2} \exp\left[-\nu \left(\frac{|x-\xi|}{h_m(\xi)}\right)^{\epsilon}\right].$$

The Lebesgue constant $\mathcal{L}_{\Xi,\sigma} = \|I_\Xi\|_{L_\infty(h_m^{-\sigma}) \to L_\infty(h_m^{-\sigma})}$ is uniformly bounded (for $\sigma \leq 2m$) and

$$|f(x) - I_\Xi f(x)| \lesssim (1 + \mathcal{L}_{\Xi,\sigma})\big(h_m(x)\big)^{\sigma} \|f\|_{C^\sigma}$$

Kernels on manifolds [H-Narcowich-Ward, '10]

$\mathbb{M}$ a compact d dimensional Riemannian manifold. If $k : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ is the reproducing kernel for $W_2^m(\mathbb{M})$, $m > d/2$

- Lagrange function is bounded in native space norm

$$\|\chi_\xi\|_{W_2^m(\mathbb{M})} \leq Cq^{d/2-m}.$$

  This is a bump estimate – compare χ_ξ to an interpolant with support in $B(\xi, q)$. Here $q := \min_{\xi \in \Xi} \mathrm{dist}(\xi, \Xi \setminus \{\xi\})$.

- Lagrange coefficients are uniformly bounded:

$$|A_{\xi,\zeta}| = |\langle \chi_\xi, \chi_\zeta \rangle_{W_2^m}| \leq Cq^{d-2m}$$

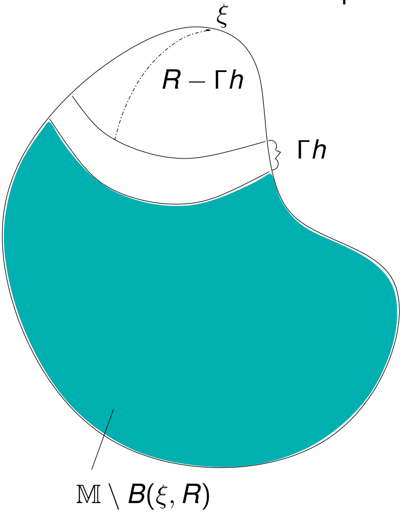
$$\longrightarrow \|(\mathrm{C}_\Xi)^{-1}\|_\infty \leq Cq^{d-2m}(\#\Xi)$$

- [De Marchi-Schaback, '10] If Ξ is sufficiently dense in $\mathbb{M}$, then a zeros lemma ensures that the Lagrange function is bounded, independent of $\#\Xi$:

$$|\chi_\xi(x)| \leq Cq^{d/2-m}h^{m-d/2} = C\rho^{m-d/2}$$

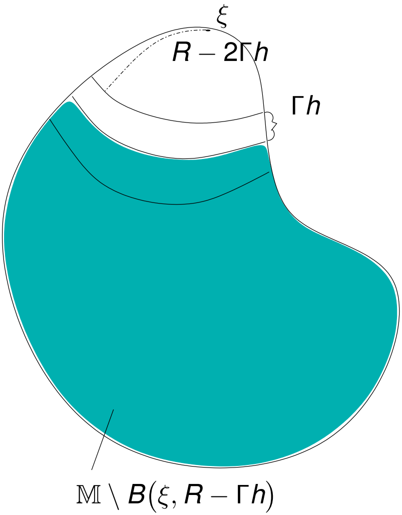
How important is it to be "boundary-free"?

We must be able to decompose $\mathbb{M}$ into annuli around ξ.



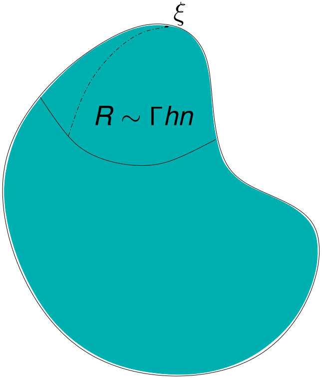
1-step energy:
$\exists \epsilon < 1$, $\Gamma > 0$, (depending only on m and $\mathbb{M}$) so that

$$\|\chi_\xi\|_{W_2^m(\mathbb{M} \setminus B(\xi,R))} \leq \epsilon \|\chi_\xi\|_{W_2^m(\mathbb{M} \setminus B(\xi,R-\Gamma h))}$$



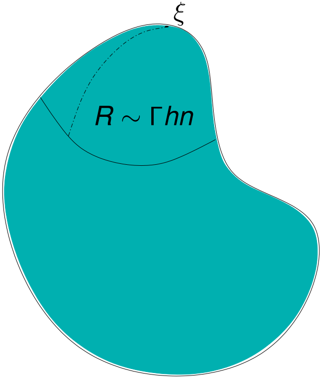
Bulk Chasing

$$
\begin{aligned}
&\|\chi_\xi\|_{W_2^m(\mathbb{M}\setminus B(\xi,R))} \\
&\le \epsilon \|\chi_\xi\|_{W_2^m(\mathbb{M}\setminus B(\xi,R-\Gamma h))} \\
&\le \epsilon^2 \|\chi_\xi\|_{W_2^m(\mathbb{M}\setminus B(\xi,R-2\Gamma h))} \\
&\vdots \\
&\le \epsilon^n \|\chi_\xi\|_{W_2^m(\mathbb{M})}
\end{aligned}
$$



Energy estimate

$$\|\chi_\xi\|_{W_2^m(\mathbb{M}\setminus B(\xi,R))} \lesssim e^{-\nu(\frac{R}{h})}\|\chi_\xi\|_{W_2^m(\mathbb{M})} \lesssim e^{-\nu(\frac{R}{h})}\, q^{d/2-m}$$



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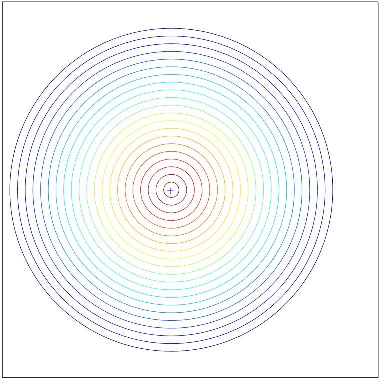
Pointwise estimate

$$|\chi_\xi(x)| \lesssim \left(\frac{h}{q}\right)^{m-d/2} e^{-\nu\left(\frac{\mathrm{dist}(\xi,x)}{h}\right)}$$

How important is it to be “boundary-free”?

Consider $\Omega = [0,1]^2$:

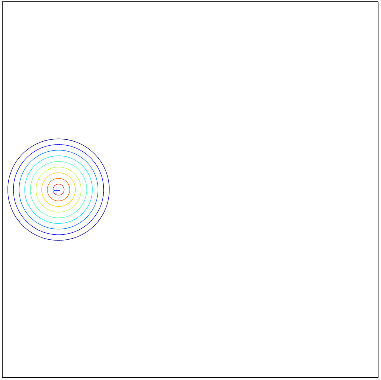
There are 22 annuli around the center (.45, .5)



How important is it to be “boundary-free”?

Consider $\Omega = [0,1]^2$: This argument breaks down for centers near the boundary where we can place fewer annuli.

There are 7 annuli around the center (.15, .5)

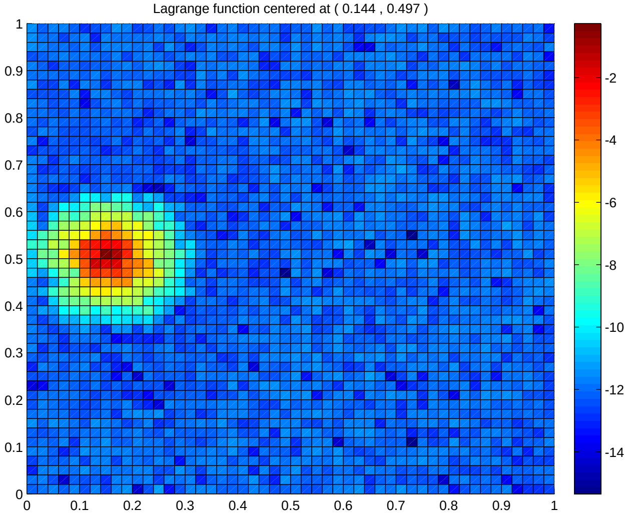


For $R < \mathrm{dist}(\xi, \partial\Omega)$

$$\|\chi_\xi\|_{W_2^m(\Omega \setminus B(x,R))} \leq C q^{d/2-m} e^{-\nu \frac{R}{h}}.$$

Boundary effects for Lagrange functions

A Lagrange function centered in the interior of $[0, 1]^2$.



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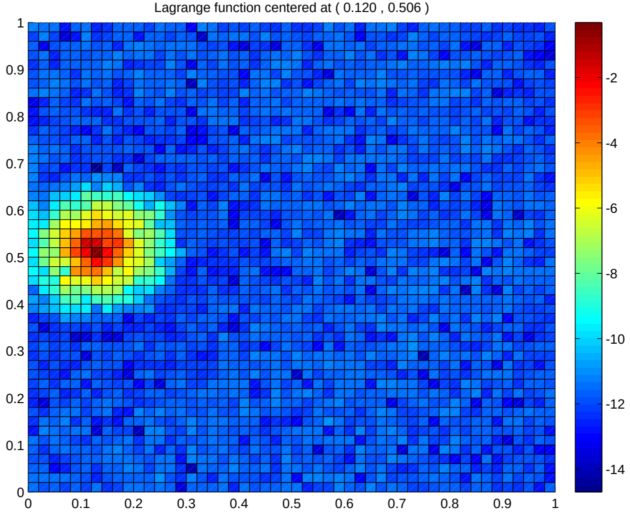

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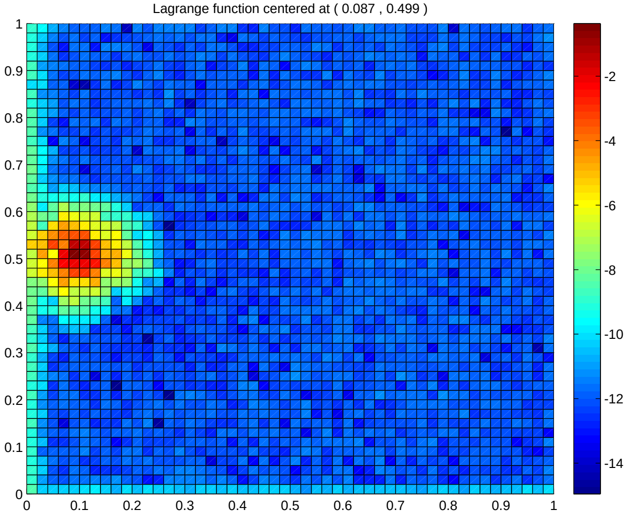

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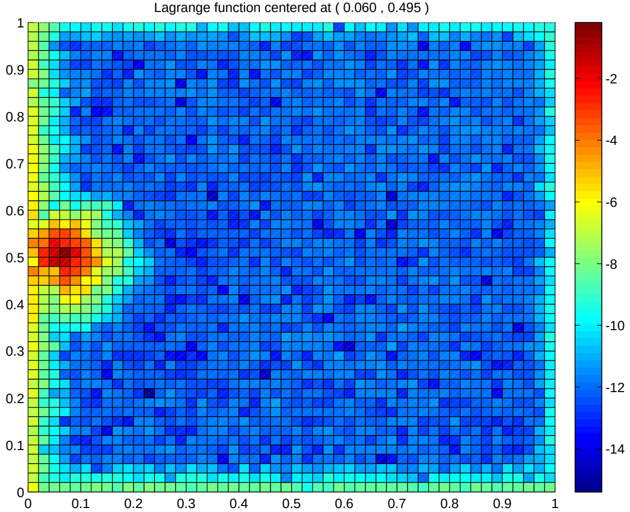

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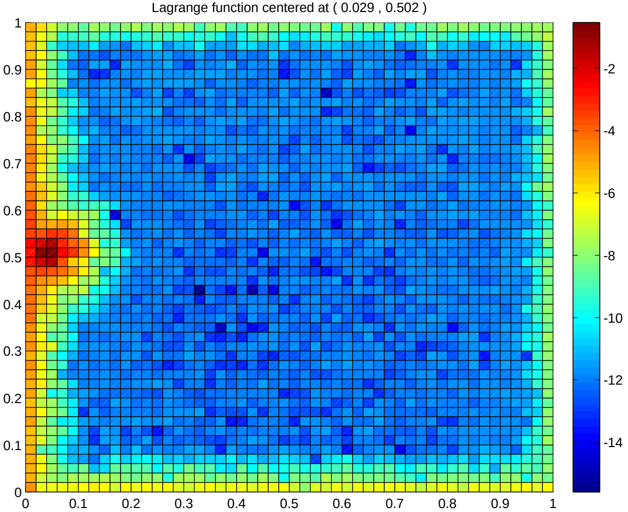

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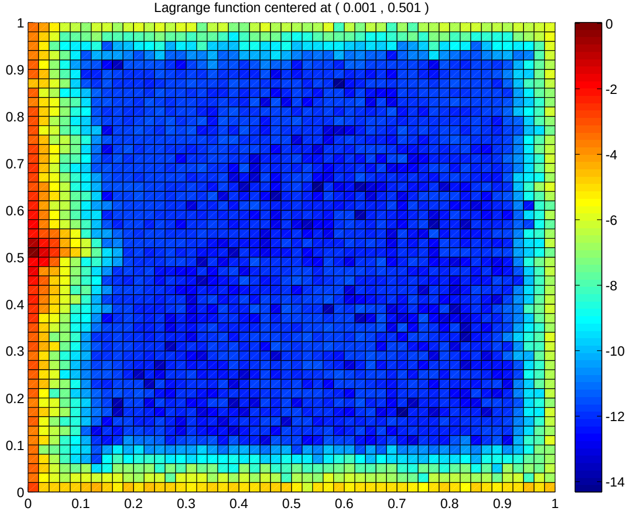
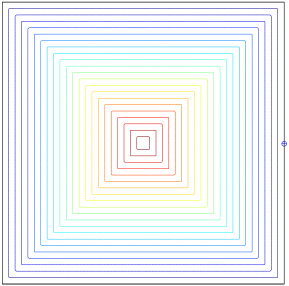


How important is it to be “boundary-free”?

For compact $\Omega \subset \mathbb{R}^d$, and $0 \leq R$, $\Omega_R := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) \geq R\}$.

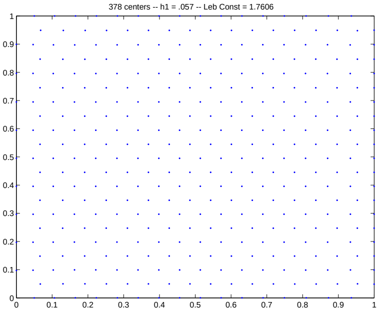


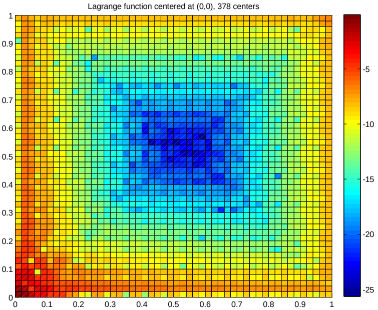
There exist positive constants C, h_0 and ν depending only on $\partial\Omega$ and m so that for $h < h_0$ and $\mathrm{dist}(\xi, \partial\Omega) \leq R$, we have

$$\|\chi_\xi\|_{W_2^m(\Omega_R)} \leq C q^{d/2-m} e^{-\nu \frac{R-\mathrm{dist}(\xi,\partial\Omega)}{h}}.$$

Question: For $\xi \in \partial\Omega$, does $|\chi_\xi(x)|$ decay along boundary?

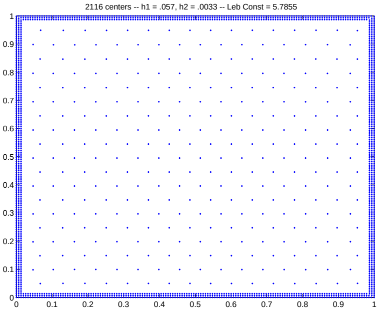
A final experiment

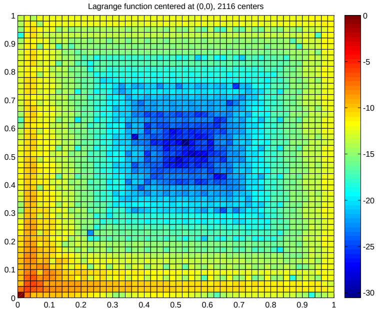




378 relatively equispaced centers, and the (log-scaled) Lagrange function centered at the origin.

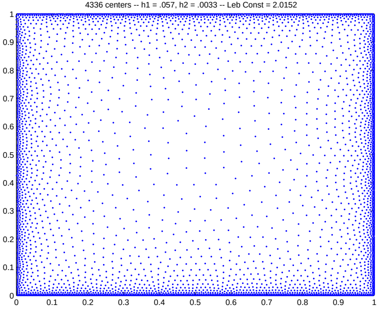
A final experiment

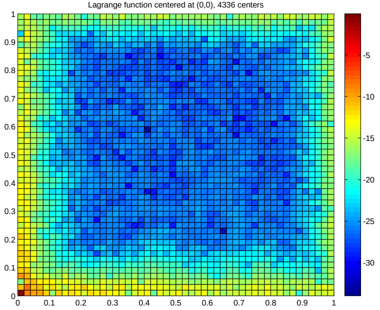




Adding three rings of centers near the boundary, with spacing $h_2 = .0033$.

A final experiment





Replacing these with nicely varying centers (created with DistMesh), having $.0033 \le h(x) \le .057$, and satisfying

$$h(x) \lesssim h(y)\left(1 + \frac{|x-y|}{h(y)}\right)^{7/12}$$

(i.e., $\epsilon = 5/12$).