Math 244
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Final Exam
Name SOLUTIONS

Instructions: Write legibly. Indicate your answers clearly. Show ALL your work. No calculators, phones, or other electronic devices.

Problem	Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
8	15	
9	15	
10	10	
Total	150	

1. (10 points each) Give a parametrization for the following surfaces. Don't forget to include the parameter domain.

(a) S is part of the plane $z = x + 3$ that is inside the cylinder $x^2 + y^2 = 1$.

Cylindrical Coordinates : $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\boxed{\begin{gathered}\vec{r}(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ r\cos\theta + 3\rangle \\ 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\end{gathered}}$$

(b) S the surface of the sphere $x^2 + y^2 + z^2 = 1$ that lies inside the cone $z = \sqrt{3(x^2 + y^2)}$.

Spherical Coordinates: $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$

$z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}$ ← sphere

$z = \sqrt{3(x^2+y^2)} = \sqrt{3r^2}$ ← Cone

Set these equal: $3r^2 = 1 - r^2 \Rightarrow 4r^2 = 1 \Rightarrow r = \frac{1}{2}$

Since $r = \rho\sin\phi$ and $\rho = 1$ (sphere $x^2+y^2+z^2=1$), we have

$r = \frac{1}{2} \Rightarrow \sin\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{6}$.

$$\boxed{\begin{gathered}\vec{r}(\theta,\phi) = \langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle \\ 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{6}\end{gathered}}$$

2. Let S be the surface of the helicoid parametrized by

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle \quad 0 \le u \le 1, \quad 0 \le v \le \pi.$$

Compute the surface area of S

$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$, $\vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$

$\vec{r}_u \times \vec{r}_v = \langle \sin v, -\cos v, u\rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{1+u^2}$

$$\text{Surface Area} = \iint_S d\sigma = \int_0^1 \int_0^\pi \sqrt{1+u^2}\, dv\, du$$

$$= \pi \int_0^1 \sqrt{1+u^2}\, du$$

$u = \tan\theta$
$du = \sec^2\theta\, d\theta$

u	0	1
θ	0	$\frac{\pi}{4}$

$$= \pi \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$$

$$= \pi\left[\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta|\right]$$

$$= \boxed{\frac{\pi}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right]}$$

3. Evaluate the surface integral

$$\iint_S \frac{x^2}{z}\, d\sigma,$$

where S is part of the paraboloid $z = x^2 + y^2$ between $z = 2$ and $z = 6$.

$z = x^2 + y^2 \Rightarrow z - x^2 - y^2 = 0$. Let $G(x,y,z) = z - x^2 - y^2$.

Then S is given by $G(x,y,z) = 0$.

$\vec{\nabla} G = \langle -2x, -2y, 1 \rangle$

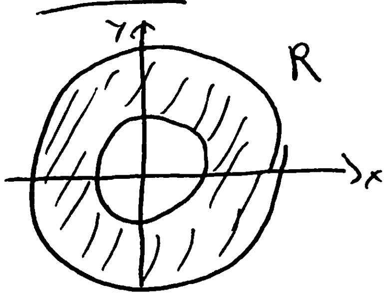
$|\vec{\nabla} G| = \sqrt{4x^2 + 4y^2 + 1}$

The shadow of S lies on the xy-plane, so take $\vec{p} = \vec{k}$

Then $|\vec{\nabla} G \cdot \vec{k}| = |1| = 1$

$\therefore \quad d\sigma = \dfrac{|\vec{\nabla} G|}{|\vec{\nabla} G \cdot \vec{k}|}\, dA = \sqrt{4x^2 + 4y^2 + 1}\, dA$

Shadow



Polar: $\sqrt{2} \le r \le \sqrt{6}$

$0 \le \theta \le 2\pi$

Thus, $\displaystyle \iint_S \frac{x^2}{z}\, d\sigma = \iint_R \frac{x^2}{x^2+y^2} \sqrt{4x^2+4y^2+1}\, dA$

$$= \int_0^{2\pi} \int_{\sqrt{2}}^{\sqrt{2}} \frac{r^2 \cos^2\theta}{r^2} \cdot \sqrt{4r^2+1} \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\, d\theta \cdot \int_{\sqrt{2}}^{\sqrt{6}} r\sqrt{4r^2+1}\, dr \leftarrow u = 4r^2+1$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + \cos(2\theta))\, d\theta \cdot \frac{1}{8}\int_9^{25} u^{\frac{1}{2}}\, du$$

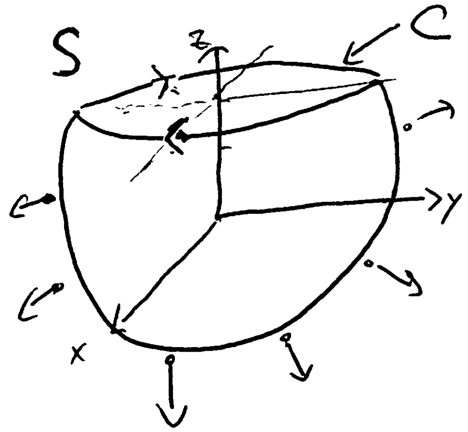
$$= \frac{1}{16}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{2\pi} \cdot \left[\frac{2}{3} u^{3/2}\right]_{\sqrt{2}}^{\sqrt{6}}$$

$$= \boxed{\frac{49\pi}{6}}$$

4. Evaluate

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, d\sigma,$$

where $\vec{F} = \langle yz, 3xz, z^2 \rangle$ and S is part of the surface of the sphere $x^2 + y^2 + z^2 = 16$ that lies below the plane $z = 2$, and oriented outward.



Note: Given the orientation of S, the orientation of the curve C is clockwise!

We parametrize C with clockwise orientation:

$$\vec{r}(t) = \langle 2\sqrt{3}\cos t, -2\sqrt{3}\sin t, 2 \rangle, \quad 0 \le t \le 2\pi$$

$$\vec{v}(t) = \langle -2\sqrt{3}\sin t, -2\sqrt{3}\cos t, 0 \rangle$$

$z = 2$: $x^2 + y^2 + 4 = 16$

$\Rightarrow x^2 + y^2 = 12$

$\Rightarrow r = \sqrt{12} = 2\sqrt{3}$

$$\vec{F}(\vec{r}(t)) = \langle -4\sqrt{3}\sin t, 12\sqrt{3}\cos t, 4 \rangle$$

$$\vec{F} \cdot d\vec{r} = \left(24\sin^2 t - 72\cos^2 t\right) dt = 24(\sin^2 t - 3\cos^2 t)\, dt$$
$$= 24(1 - 4\cos^2 t)\, dt$$

By Stokes' Theorem,

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 24\left(1 - 4\cos^2 t\right) dt$$

$$= 24 \int_0^{2\pi} \left(1 - 2(1 + \cos(2t))\right) dt$$

$$= 24 \int_0^{2\pi} \left(-1 - 2\cos(2t)\right) dt$$

$$= 24 \left[-t - \sin(2t)\right]_0^{2\pi}$$

$$= \boxed{-48\pi}$$

5. Let $\vec{F} = \langle 3y^2z^3, 9x^2yz^2, -4xy\rangle$. Compute the outward flux of $\vec{F}$ across the surface S of the cube $-1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1$.

$$\operatorname{div} \vec{F} = \vec{\nabla}\cdot\vec{F} = 0 + 9x^2z^2 + 0 = 9x^2z^2$$

By the Divergence Theorem,

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_R \vec{\nabla}\cdot\vec{F}\, dV$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 9x^2z^2\, dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left[\tfrac{9}{3}x^2z^3\right]_{-1}^{1} dy\,dx$$

$$= 3\int_{-1}^{1}\int_{-1}^{1} 2x^2\, dy\,dx$$

$$= 3\int_{-1}^{1} 2x^2y\Big|_{-1}^{1} dx$$

$$= 3\int_{-1}^{1} 4x^2\, dx$$

$$= 3\left[\tfrac{4}{3}x^3\right]_{-1}^{1}$$

$$= 3\cdot\frac{8}{3}$$

$$= 8$$

6. A wire shaped like a helix is parametrized by

$$\vec{r}(t) = \langle 2\sin t, 2\cos t, 3t \rangle, \qquad 0 \le t \le 2\pi$$

If the wire has constant density δ, find the mass and center of mass of the wire.

$\vec{v}(t) = \langle 2\cos t, -2\sin t, 3 \rangle$, $|\vec{v}(t)| = \sqrt{4\cos^2 t + 4\sin^2 t + 9}$
$= \sqrt{13}$

$$M = \int_C \delta\, ds = \int_0^{2\pi} \delta\sqrt{13}\, dt = \boxed{2\pi\sqrt{13}\,\delta}$$

$$M_{yz} = \int_C x\,\delta\, ds = \int_0^{2\pi} 2\sin t \cdot \delta\sqrt{13}\, dt = 2\sqrt{13}\,\delta \cdot (-\cos t)\Big|_0^{2\pi} = 0 \Rightarrow \bar{x} = 0$$

$$M_{xz} = \int_C y\,\delta\, ds = \int_0^{2\pi} 2\cos t \cdot \delta\sqrt{13}\, dt = 2\sqrt{13}\,\delta(\sin t)\Big|_0^{2\pi} = 0 \Rightarrow \bar{y} = 0$$

$$M_{xy} = \int_C z\,\delta\, ds = \int_0^{2\pi} 3t\,\delta\sqrt{13}\, dt = 3\sqrt{13}\,\delta \cdot \frac{t^2}{2}\Big|_0^{2\pi} = 12\sqrt{13}\,\delta\,\pi^2$$

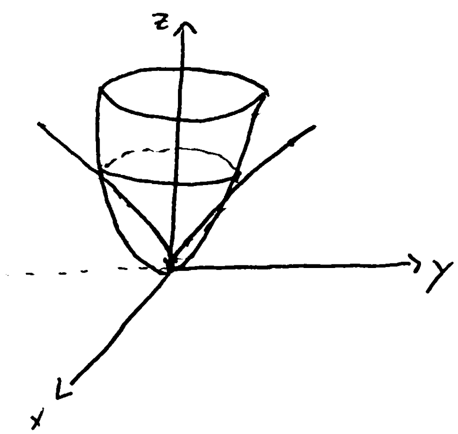
$$\Rightarrow \bar{z} = \frac{12\cancel{\sqrt{13}}\cancel{\delta}\,\pi^2}{2\cancel{\sqrt{13}}\cancel{\delta}\cancel{\pi}} = 6\pi$$

$$\therefore \boxed{C.M = (0, 0, 6\pi)}$$

7. Evaluate the triple integral

$$\iiint_R dV,$$

Where R is the region in space above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$.



In cylindrical coordinates:

paraboloid : $z = r^2$

halfcone : $z = r$

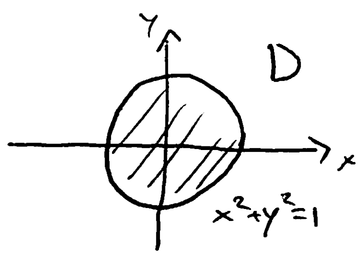
These intersect when $r^2 = r$

$\Rightarrow r^2 - r = 0$

$\Rightarrow r(r-1) = 0$

$\Rightarrow r = 0, \; r = 1$

$\therefore$ shadow: Disk of radius 1.



$\rightarrow \; 0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\therefore \iiint_R dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 rz\Big|_{r^2}^{r}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^2 - r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{3} - \frac{1}{4}\right)d\theta = \frac{1}{12}\cdot 2\pi = \boxed{\frac{\pi}{6}}$$

8. Consider the double integral

$$\iint_R \frac{x-y}{x+y}\, dA,$$

where R is the square region in the plane with vertices $(0,2),(1,1),(2,2),$ and $(1,3)$. Now let $u = x - y$ and $v = x + y$ be a coordinate transformation. Rewrite (but do not evaluate) the above double integral in terms of u and v (Include the new limits of integration).

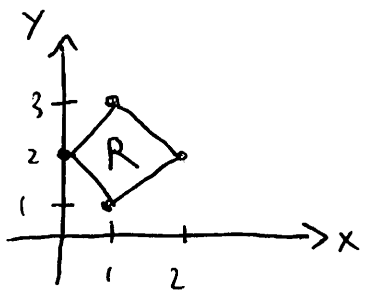
$$u = x - y \Rightarrow \underline{x = u + y}$$

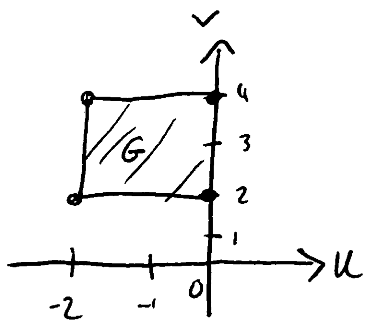
$$v = x + y = (u+y) + y = u + 2y \Rightarrow 2y = v - u \Rightarrow \boxed{y = \tfrac{1}{2}(v-u)}$$

$$x = u + y = u + \tfrac{1}{2}(v-u) = \tfrac{1}{2}(v+u) \rightarrow \boxed{x = \tfrac{1}{2}(v+u)}$$

(Use these equations to transform the region R and to compute $J(u,v)$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \underline{\underline{\frac{1}{2}}}$$





$-2 \le u \le 0$

$2 \le v \le 4$

$$\therefore \boxed{\iint_R \frac{x-y}{x+y}\, dA = \int_{-2}^{0} \int_{2}^{4} \frac{u}{v} \cdot \frac{1}{2}\, dv\, du}$$

9. Let $\vec{F} = \langle 2xz + y^2, 2xy, x^2 + 3z^2 \rangle$. Show that $\vec{F}$ is conservative and find a potential function for it.

$$\text{Curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz+y^2 & 2xy & x^2+3z^2 \end{vmatrix} = \langle 0, 0, 0 \rangle$$

$\therefore$ $\vec{F}$ is conservative.

Let $f(x,y,z)$ be such that $\vec{\nabla} f = \vec{F}$

ie: $\frac{\partial f}{\partial x} = 2xz+y^2, \quad \frac{\partial f}{\partial y} = 2xy, \quad \frac{\partial f}{\partial z} = 3z^2$

$$\frac{\partial f}{\partial x} = 2xz + y^2 \Rightarrow f(x,y,z) = x^2 z + xy^2 + g(y,z)$$

$$\frac{\partial f}{\partial y} = 2xy + \frac{\partial g}{\partial y} = 2xy \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$$

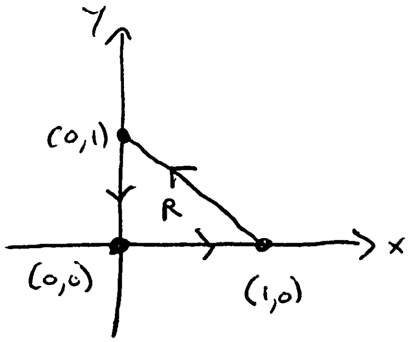
$$\therefore f(x,y,z) = x^2 z + xy^2 + h(z)$$

$$\frac{\partial f}{\partial z} = x^2 + h'(z) = x^2 + 3z^2 \Rightarrow h'(z) = 3z^2 \Rightarrow h(z) = z^3 + C$$

$$\therefore \boxed{f(x,y,z) = x^2 z + xy^2 + z^3 + C}$$

10. Use Green's Theorem to find the work done by the force $\vec{F} = \langle x^2 + xy, xy^2 \rangle$ in moving a particle along the triangular region from $(0,0)$ to $(1,0)$ to $(0,1)$ and back to $(0,0)$.

Work $= \int_C \vec{F} \cdot d\vec{r}$, where C is the boundary of the triangle



Let R be the region enclosed by C.

$P = x^2 + xy \qquad Q = xy^2$

$\frac{\partial P}{\partial y} = x \qquad \frac{\partial Q}{\partial x} = y^2$

By Green's Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (y^2 - x)\, dA$$

$$= \int_0^1 \int_0^{1-x} (y^2 - x)\, dy\, dx$$

$$= \int_0^1 \left[\frac{y^3}{3} - xy \right]_0^{1-x} dx$$

$$= \int_0^1 \left(\frac{(1-x)^3}{3} - x(1-x) \right) dx$$

$$= \int_0^1 \left(-\frac{x^3}{3} + 2x^2 - 2x + \frac{1}{3} \right) dx$$

$$= \left[-\frac{x^4}{12} + \frac{2x^3}{3} - x^2 + \frac{x}{3} \right]_0^1 = -\frac{1}{12} + \frac{2}{3} - 1 + \frac{1}{3} = \frac{-1+8-12+4}{12} = \boxed{-\frac{1}{12}}$$

Scratch Paper

Formula Sheet

Trigonometric Identities

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta)), \qquad \sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$$

Coordinate Conversion Formulas

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$

$$y = r\sin\theta = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi, \qquad r = \rho\sin\phi, \qquad x^2+y^2+z^2 = \rho^2$$

$$dV = dx\,dy\,dz = dz\,r\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Jacobian

If $x = x(u,v)$ and $y = y(u,v)$ is a coordinate transformation, then the *Jacobian* is given by

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Moments and Center of Mass

Quantity	Triple Integral	Line Integral	Surface Integral
M	$\iiint_R \delta\,dV$	$\int_C \delta\,ds$	$\iint_S \delta\,d\sigma$
M_{yz}	$\iiint_R x\,\delta\,dV$	$\int_C x\,\delta\,ds$	$\iint_S x\,\delta\,d\sigma$
M_{xz}	$\iiint_R y\,\delta\,dV$	$\int_C y\,\delta\,ds$	$\iint_S y\,\delta\,d\sigma$
M_{xy}	$\iiint_R z\,\delta\,dV$	$\int_C z\,\delta\,ds$	$\iint_S z\,\delta\,d\sigma$
I_x	$\iiint_R (y^2+z^2)\delta\,dV$	$\int_C (y^2+z^2)\delta\,ds$	$\iint_S (y^2+z^2)\delta\,d\sigma$
I_y	$\iiint_R (x^2+z^2)\delta\,dV$	$\int_C (x^2+z^2)\delta\,ds$	$\iint_S (x^2+z^2)\delta\,d\sigma$
I_z	$\iiint_R (x^2+y^2)\delta\,dV$	$\int_C (x^2+y^2)\delta\,ds$	$\iint_S (x^2+y^2)\delta\,d\sigma$

Center of Mass: $(\overline{x},\overline{y},\overline{z})$, where $\overline{x} = \frac{M_{yz}}{M}, \overline{y} = \frac{M_{xz}}{M}, \overline{z} = \frac{M_{xy}}{M}$.

Green's Theorem

Let C be a positively oriented (counterclockwise) piecewise-smooth, simple closed curve in the plane, and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

If $\vec{F} = P\vec{i} + Q\vec{j}$ is a vector field, then

Outward Flux of $\vec{F}$ *Across* C:

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

Counterclockwise Circulation of $\vec{F}$ *Across* C:

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Curl and Divergence

If $\vec{F} = P\vec{i}+Q\vec{j}+R\vec{k}$ is a vector field and $\vec{\nabla}$ is the del operator given by $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}$, then

$$\text{curl } \vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}, \qquad \text{div } \vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Stokes' Theorem

Let $\vec{F}$ be a vector field and let S be a piecewise-smooth, oriented surface with boundary curve C. Then

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r}.$$

Divergence Theorem

Let $\vec{F}$ be a vector field and let S be a closed, piecewise-smooth, oriented surface enclosing a region R in space. Then

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_R \vec{\nabla}\cdot\vec{F}\,dV.$$