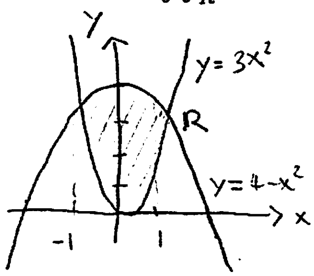
Math 244
Jamal Hassan
Practice Midterm 1
Name ________________

Instructions: Write legibly. Indicate your answers clearly. Show ALL your work. No calculators, phones, or other electronic devices.

Problem	Points	Score
1	16	
2	15	
3	20	
4	15	
5	16	
6	18	
Total	100	

1. (8 points each) Evaluate the following integrals.

a) $\iint_R \frac{e^x}{1-x}\, dA$, where R is the region bounded by the parabolas $y = 3x^2$ and $y = 4 - x^2$.



$3x^2 = 4 - x^2$

$\Rightarrow 4x^2 = 4$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

$$\int_{-1}^{1}\int_{3x^2}^{4-4x^2} \frac{e^x}{1-x}\, dy\, dx$$

$$= \int_{-1}^{1} \frac{e^x}{1-x}\cdot(4-4x^2-3x^2)\, dx$$

$$= \int_{-1}^{1} \frac{e^x}{1-x}\, 4(1-x)(1+x)\, dx$$

$$= 4\int_{-1}^{1} (1+x)e^x\, dx$$

int. by parts

$u = 1+x \quad v = e^x$

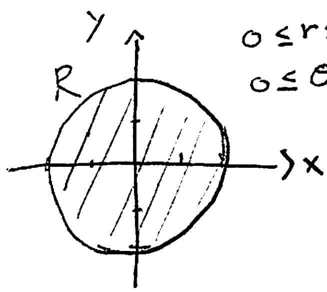
$du = dx \quad dv = e^x\, dx$

$$= 4\left[(1+x)e^x\Big|_{-1}^{1} - \int_{-1}^{1} e^x\, dx\right]$$

$$= 4\left[(1+x)e^x - e^x\right]_{-1}^{1} = 4xe^x\Big|_{-1}^{1} = \boxed{4\left(e+\frac{1}{e}\right)}$$

b) $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \ln(x^2+y^2+2)\, dA$

Convert to polar:



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^2 \ln(r^2+2)\, r\, dr\, d\theta$$

$$= 2\pi \int_0^2 \ln(r^2+2)\, r\, dr$$

u-sub

$u = r^2 + 2$

$\frac{1}{2} du = r\, dr$

r	0	2
u	2	6

$$= 2\pi \int_2^6 \frac{1}{2}\ln(u)\, du$$

int by parts

$s = \ln(u) \quad t = u$

$ds = \frac{1}{u} du \quad dt = du$

$$= \pi\left[u\ln(u)\Big|_2^6 - \int_2^6 du\right]$$

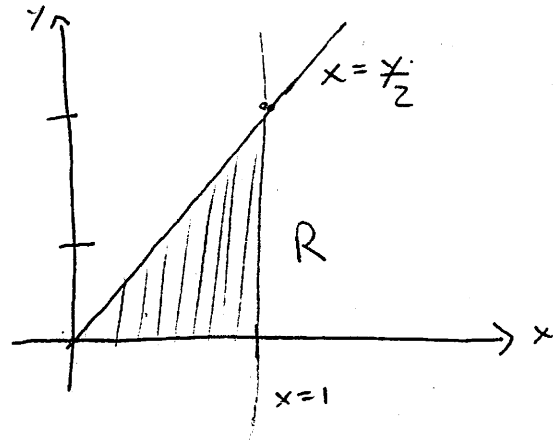
$$= \pi\left[u\ln(u) - u\right]_2^6$$

$$= \pi\left[6\ln 6 - 6 - 2\ln(2) + 2\right]$$

$$= 2\pi\left[3\ln 6 - \ln(2) - 2\right] = \boxed{2\pi\left[\ln(108) - 2\right]}$$

2. Consider the integral $\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$.

a) (5 points) Sketch the region R of integration on the xy-plane.



b) (10 points) Evaluate the integral.

switch order of integration:

$$\int_{x=0}^{1} \int_{y=0}^{2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 e^{x^2} \cdot 2x\, dx \quad \leftarrow$$

u-sub

$u = x^2$

$du = 2x\, dx$

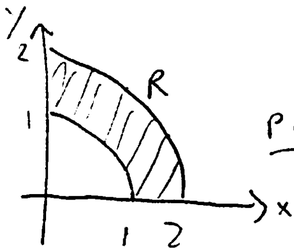
x	0	1
u	0	1

$$= \int_0^1 e^u\, du$$

$$= \boxed{e-1}$$

3. A lamina occupies the region R in the first quadrant bounded by the equations $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and has variable density given by $\delta(x, y) = x + y$.

a) (5 points) Find the mass of the lamina.



Polar

$0 \le \theta \le \frac{\pi}{2}$

$1 \le r \le 2$

$$M = \iint_R (x+y)\,dA = \int_0^{\frac{\pi}{2}} \int_1^2 (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 r^2 (\cos\theta + \sin\theta)\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}(8-1)(\cos\theta + \sin\theta)\,d\theta$$

$$= \frac{7}{3}\left[\sin\theta - \cos\theta\right]_0^{\frac{\pi}{2}} = \frac{7}{3}\left[(1-0) - (0-1)\right] = \boxed{\frac{14}{3}}$$

b) (10 points) Find the center of mass of the lamina.

$$M_y = \iint_R x(x+y)\,dA = \int_0^{\frac{\pi}{2}} \int_1^2 r^3(\cos^2\theta + \cos\theta\sin\theta)\,dr\,d\theta$$

$$= \frac{r^4}{4}\Big|_1^2 \int_0^{\frac{\pi}{2}} \Big(\frac{1}{2} + \frac{1}{2}\cos(2\theta) + \underbrace{\cos\theta\sin\theta}_{u\text{-sub}}\Big)\,d\theta$$

$$= \frac{15}{4}\left[\frac{1}{2}\theta + \cancel{\frac{\sin(2\theta)}{4}} + \frac{\sin^2(\theta)}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{15}{4}\left(\frac{\pi}{4} + \frac{1}{2}\right) \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{3}{14}\cdot\frac{15}{4}\left(\frac{\pi}{4}+\frac{1}{2}\right) = \boxed{\frac{45}{56}\left(\frac{\pi}{4}+\frac{1}{2}\right)}$$

$$M_x = \iint_R y(x+y)\,dA = \int_0^{\frac{\pi}{2}} \int_1^2 r^3(\sin\theta\cos\theta + \sin^2\theta)\,dr\,d\theta$$

$$= \frac{15}{4}\left[\frac{\sin^2(\theta)}{2} + \frac{\theta}{2} - \cancel{\frac{\sin(2\theta)}{4}}\right]_0^{\frac{\pi}{2}} = \frac{15}{4}\left(\frac{1}{2} + \frac{\pi}{4}\right) \rightarrow \bar{y} = \frac{M_x}{M} = \boxed{\frac{45}{56}\left(\frac{\pi}{4}+\frac{1}{2}\right)}$$

c) (5 points) Find the lamina's moment of inertia about the y-axis.

$$I_y = \iint_R x^2(x+y)\,dA = \int_0^{\frac{\pi}{2}} \int_1^2 r^4(\cos^3\theta + \cos^2\theta\sin\theta)\,dr\,d\theta$$

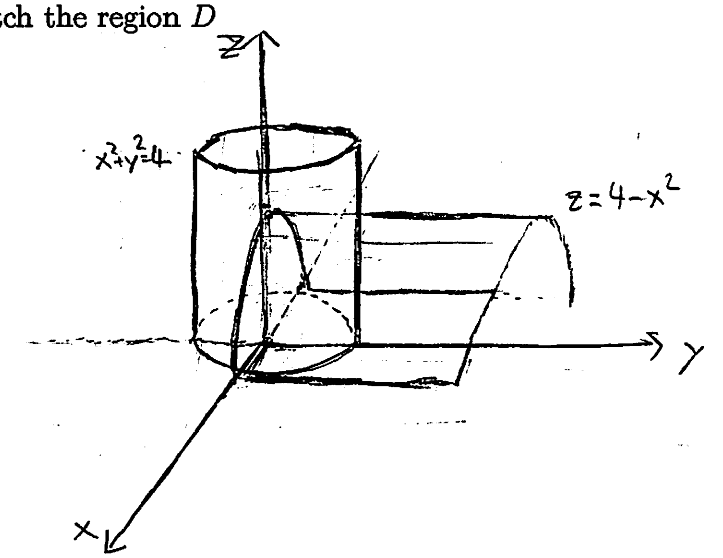
$$= \frac{r^5}{5}\Big|_1^2 \int_0^{\frac{\pi}{2}} (\cos^3\theta + \cos^2\theta\sin\theta)\,d\theta$$

$$= \frac{31}{5}\int_0^{\frac{\pi}{2}} \left[(1-\sin^2\theta)\cos\theta + \cos^2\theta\sin\theta\right] d\theta = \frac{31}{5}\left[\int_0^1 (1-u^2)\,du + \int_0^1 u^2\,du\right]$$

$$= \frac{31}{5}\left[u + \frac{2}{3}u^3\right]_0^1 = \frac{31}{5}\left(1 + \frac{2}{3}\right) = \boxed{\frac{31}{3}}$$

4. Consider the region D in space bounded above by the curve $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the xy-plane.

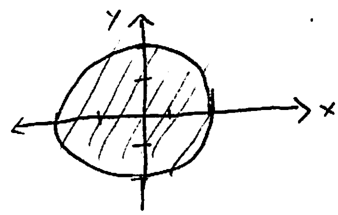
a) (5 points) Sketch the region D



b) (10 points) Set up and evaluate the triple integral that gives the volume of D.

The projection (shadow) of D onto the xy-plane is the circle

$x^2+y^2=4$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

<u>Convert to cylindrical</u>: $z = 4 - x^2 = 4 - r^2\cos^2\theta$

$$\text{Vol} = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{4-r^2\cos^2\theta} r \, dz \, dr \, d\theta$$

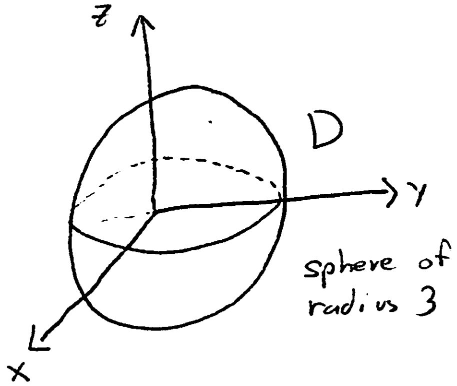
$$= \int_0^{2\pi} \int_0^2 (4r - r^3\cos^2\theta) \, dr \, d\theta$$

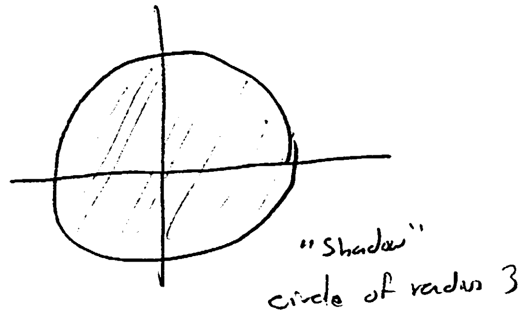
$$= \int_0^{2\pi} \left[2r^2 - \frac{r^4\cos^2\theta}{4} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} (8 - 4\cos^2\theta) \, d\theta = \int_0^{2\pi} (8 - 2 - 2\cos(2\theta)) \, d\theta$$

$$= \Big[6\theta - \sin(2\theta) \Big]_0^{2\pi} = \boxed{12\pi}$$

5 (16 points) Find the average value of the function $F(x,y,z) = \sqrt{x^2+y^2+z^2}$ over the sphere of radius 3 centered at the origin.





Use spherical coordinates

$$0 \le \rho \le 3$$
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$
$$F(\rho,\phi,\theta) = \rho$$

Volume of $D = \int_0^{2\pi}\int_0^{\pi}\int_0^{3} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$= \ldots$

or just use formula for a sphere:

$\frac{4}{3}\pi \cdot 3^3 = \underline{36\pi}$

$$\iiint_D F\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{3} \rho\cdot\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^4}{4}\Big|_0^3 \sin\phi\, d\phi\, d\theta$$

$$\frac{81}{4}\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta$$

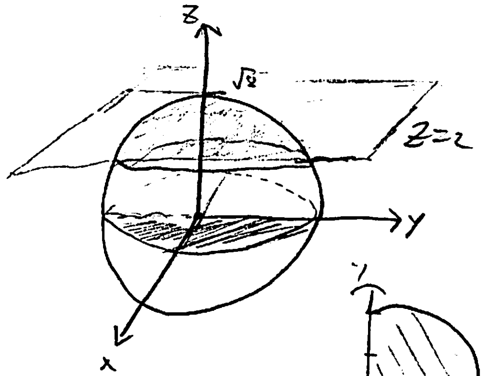
$$= \frac{81}{4}\int_0^{2\pi} -\cos(\phi)\Big|_0^{\pi} d\theta$$

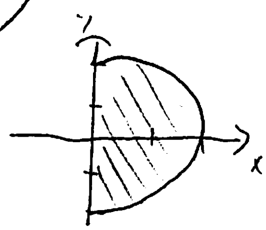
$$= \frac{81}{4}\cdot 0 = 0$$

$\therefore$ Average value $= \frac{0}{36\pi} = \boxed{0}$

6. Consider the triple integral $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_2^{\sqrt{8-x^2-y^2}} z \; dz \; dy \; dx$ given in rectangular coordinates.

a) (5 points) Rewrite an equivalent triple integral in cylindrical coordinates.





$$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \int_2^{\sqrt{8-r^2}} z r \; dz\, dr\, d\theta}$$

b) (5 points) Rewrite an equivalent triple integral in spherical coordinates.

$\rho = \sqrt{8} = 2\sqrt{2}$

$\hookrightarrow z = 2 \Rightarrow \rho\cos\phi = 2 \Rightarrow \boxed{\rho = 2\sec\theta}$

$\Rightarrow 2\sqrt{2}\cos\phi = 2$

$\Rightarrow \cos\phi = \frac{1}{\sqrt{2}}$

$\Rightarrow \boxed{\phi = \frac{\pi}{4}}$

$$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_{2\sec\theta}^{2\sqrt{2}} \rho\cos\phi \, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$$

c) (8 points) Evaluate the triple integral using any coordinate system.

From part (a): $\text{Vol} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \frac{1}{2}\left[8 - r^2 - 4\right] r \, dr\, d\theta$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 (4r - r^3)\, dr\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$

$$= \frac{1}{2}(8-4)\cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{2}(4)\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \boxed{2\pi}$$