Math 244
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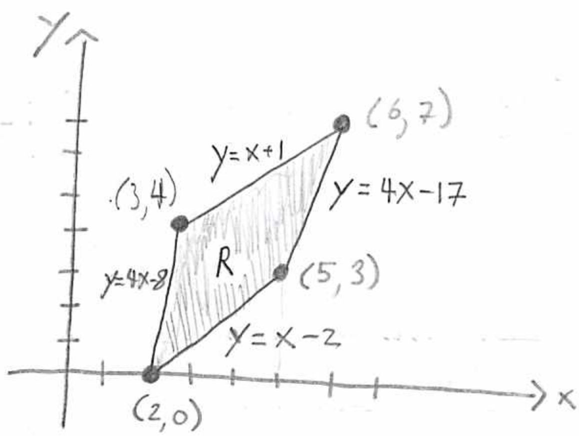
Practice Midterm 2
Name ______________

Instructions: Write legibly. Indicate your answers clearly. Show ALL your work. No calculators, phones, or other electronic devices.

Problem	Points	Score
1	25	
2	15	
3	15	
4	20	
5	15	
6	10	
Total	100	

1. Consider the integral $\iint_R (6x-3y)\,dA$, where R is the parallelogram with vertices $(2,0)$, $(5,3)$, $(6,7)$, and $(3,4)$.

(a) (5 points) Sketch the region R in the xy-plane, and find the equations that make up its boundary.



L_1: $m = \frac{3}{5-2} = 1 \quad y = x-2$

L_2: $m = \frac{7-3}{6-5} = 4, \quad y-3 = 4x-20$
$y = 4x-17$

L_3: $m = \frac{7-4}{6-3} = 1, \quad y-4 = x-3$
$y = x+1$

L_4: $m = \frac{4}{3-2} = 4, \quad y = 4(x-2)$
$y = 4x-8$

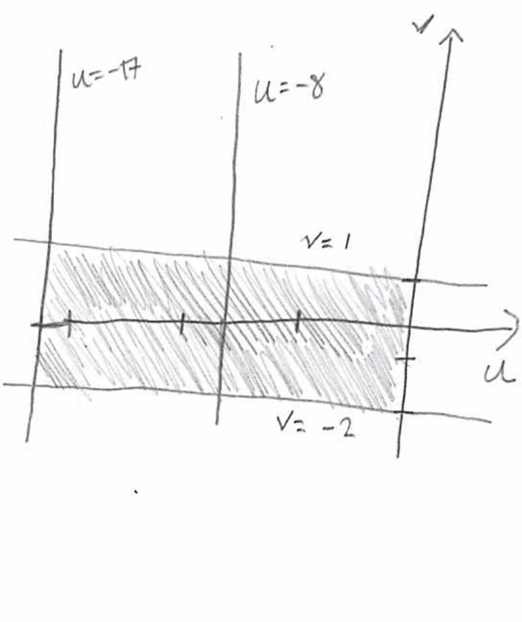
(b) (10 points) Consider the transformation $u = y-4x, v = y-x$. Sketch the region in the uv-plane that corresponds to the region R in the xy-plane via the coordinate transformation.

$u = y - 4x \Rightarrow \underline{y = u + 4x}$

$v = y - x = u + 4x - x = u + 3x \Rightarrow 3x = v - u \Rightarrow \boxed{x = \frac{1}{3}(v-u)}$

$y = u + 4x = u + \frac{4}{3}(v-u) = \frac{4}{3}v - \frac{1}{3}u \Rightarrow \boxed{y = \frac{1}{3}(4v-u)}$

xy-Boundary	uv-Boundary	Simplified
$y = x-2$	$\frac{1}{3}(4v-u) = \frac{1}{3}(v-u)-2$	$v = -2$
$y = 4x-17$	$\frac{1}{3}(4v-u) = \frac{4}{3}(v-u)-17$	$u = -17$
$y = x+1$	$\frac{1}{3}(4v-u) = \frac{1}{3}(v-u)+1$	$v = 1$
$y = 4x-8$	$\frac{1}{3}(4v-u) = \frac{4}{3}(v-u)-8$	$u = -8$



(c) (5 points) Rewrite the integral in part (a) in terms of u and v

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{vmatrix} = -\frac{4}{9} + \frac{1}{9} = -\frac{3}{9} = -\frac{1}{3}$$

$$\iint_R (6x-3y)\, dA = \int_{-17}^{-8} \int_{-2}^{1} \left(6\left(\tfrac{1}{3}(v-u)\right) - 3\left(\tfrac{1}{3}(4v-u)\right)\left|-\tfrac{1}{3}\right|\right) dv\, du$$

$$= \boxed{\int_{-17}^{-8} \int_{-2}^{1} \frac{1}{3}(-2v-u)\, dv\, du}$$

(d) (5 points) Evaluate the integral from part (c).

$$-\frac{1}{3}\int_{-17}^{-8}\int_{-2}^{1} (2v+u)\, dv\, du$$

$$= -\frac{1}{3}\int_{-17}^{-8} \left[v^2 + uv\right]_{-2}^{1} du$$

$$= -\frac{1}{3}\int_{-17}^{-8} \left[(1+u)-(4-2u)\right] du$$

$$= -\frac{1}{3}\int_{-17}^{-8} (-3+3u)\, du$$

$$= \int_{-17}^{-8} (1-u)\, du$$

$$= \left[u - \frac{u^2}{2}\right]_{-17}^{-8} = (-8-32) - \left(-17 - \frac{289}{2}\right) = \boxed{\frac{243}{2}}$$

2. Let C denote the line segment in space from the point $(1,1,0)$ to the point $(2,3,-2)$.

(a) (5 points) Find a parametrization of this curve.

$x(t) = (1-t)(1) + t(2) = t+1$

$y(t) = (1-t)(1) + t(3) = 2t+1 \qquad 0 \le t \le 1$

$z(t) = (1-t)(0) + t(-2) = -2t$

$$\boxed{\begin{gathered}\vec{r}(t) = (t+1)\vec{i} + (2t+1)\vec{j} - 2t\vec{k} \\ 0 \le t \le 1\end{gathered}}$$

(b) (10 points) Evaluate the line integral

$\int_C (xy - 4z)\, ds \qquad \vec{v}(t) = \vec{i} + 2\vec{j} - 2\vec{k}. \quad |\vec{v}(t)| = \sqrt{1+4+4} = 3$

$= \int_0^1 \left((t+1)(2t+1) - 4(-2t)\right) 3\, dt$

$= 3\int_0^1 (2t^2 + 3t + 1 + 8t)\, dt$

$= 3\int_0^1 (2t^2 + 11t + 1)\, dt$

$= 3\left[\frac{2}{3}t^3 + \frac{11}{2}t^2 + t\right]_{\cancel{0}}^1$

$= 3\left(\frac{2}{3} + \frac{11}{2} + 1\right)$

$= 3\,\frac{4+33+6}{6}$

$= \boxed{\frac{43}{2}}$

3. Let $\vec{F}(x,y,z) = \frac{2xy}{z^3}\vec{i} + (2y - z^2 + \frac{x^2}{z^3})\vec{j} - (4z^3 + 2yz + \frac{3x^2y}{z^4})\vec{k}$

(a) (3 points) Show that this vector field is conservative.

$$\frac{\partial P}{\partial y} = \frac{2x}{z^3} \qquad \frac{\partial P}{\partial z} = \frac{-6xy}{z^4}$$

$$\frac{\partial Q}{\partial x} = \frac{2x}{z^3} \qquad \frac{\partial Q}{\partial z} = -2z - \frac{3x^2}{z^4}$$

$$\frac{\partial R}{\partial x} = -\frac{6xy}{z^4} \qquad \frac{\partial R}{\partial y} = -2z - \frac{3x^2}{z^4}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Conservative

(b) (9 points) Find a potential function $f(x,y,z)$ for $\vec{F}$.

$$\frac{\partial f}{\partial x} = \frac{2xy}{z^3} \Rightarrow f(x,y,z) = \frac{x^2y}{z^3} + g(y,z)$$

$$2y - z^2 + \frac{x^2}{z^3} = \frac{\partial f}{\partial y} = \frac{x^2}{z^3} + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = 2y - z^2$$

$$\Rightarrow g(y,z) = y^2 - yz^2 + h(z)$$

$$\Rightarrow f(x,y,z) = \frac{x^2y}{z^3} + y^2 - yz^2 + h(z)$$

$$-4z^3 - 2yz - \frac{3x^2y}{z^4} = \frac{\partial f}{\partial z} = -\frac{3x^2y}{z^4} - 2yz + h'(z) \Rightarrow h'(z) = -4z^3$$

$$\Rightarrow h(z) = -z^4 + C$$

$$\boxed{\therefore\ f(x,y,z) = \frac{x^2y}{z^3} + y^2 - yz^2 - z^4 + C}$$

(c) (3 points) Find the work done by $\vec{F}$ on a particle as it moves along any curve C connecting the point $(0,0,1)$ to $(3,2,1)$

$\int_{(0,0,1)}^{(3,2,1)} \vec{F}\cdot d\vec{r}$ is path independent, so equals to

$$f(3,2,1) - f(0,0,1)$$

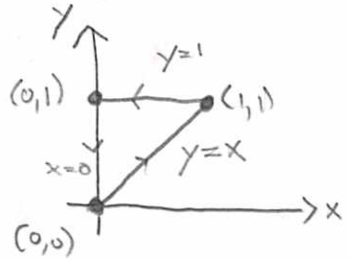
$$= (18 + 4 - 2 - 1) - (0 + 0 - 0 - 1)$$

$$= 19 + 1 = \boxed{20}$$

$2x\cos(y^2)\vec{j}$

4. Let $\vec{F}(x,y) = 2xy\vec{\imath} + \cancel{2xy\cos(y^2)\vec{\jmath}}$ denote a velocity field on the plane, and let C be the boundary of the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,1)$.

(a) (10 points) Find the counterclockwise circulation of $\vec{F}$ along C.



$$P(x,y) = 2xy \Rightarrow \frac{\partial P}{\partial x} = 2y, \quad \frac{\partial P}{\partial y} = 2x$$

$$Q(x,y) = 2x\cos(y^2) \Rightarrow \frac{\partial Q}{\partial x} = 2\cos(y^2)$$

$$\frac{\partial Q}{\partial y} = -4xy\cos(y^2)$$

$$\text{Circulation} = \oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C P\,dx + Q\,dy$$

$$\overset{\text{Apply Green's Thm}}{=} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R \left(2\cos(y^2) - 2x\right) dA$$

$u = y^2$

$du = 2y\,dy$

y	0	1
u	1	1

$$= \int_0^1 \int_0^y \left(2\cos(y^2) - 2x\right) dx\,dy$$

$$= \int_0^1 \left(2y\cos(y^2) - y^2\right) dy$$

$$= \int_0^1 \cos(u)\,du - \int_0^1 y^2\,dy$$

$$= \sin(u)\Big|_0^1 - \frac{y^3}{3}\Big|_0^1$$

$$= \boxed{\sin(1) - \frac{1}{3}}$$

(b) (10 points) Find the outward flux of $\vec{F}$ along C.

$$\text{Flux} = \oint \vec{F} \cdot \vec{n}\, ds = \oint P\,dy - Q\,dx$$

Apply Green's Th

$$= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

$$= \iint_R \left(2y - 4xy \sin(y^2) \right) dA$$

$$= \int_0^1 \int_x^1 \left(2y - 4xy \sin(y^2) \right) dy\, dx$$

$$= \int_0^1 \left(y^2 + 2x\cos(y^2) \right) \Big|_{y=x}^{1} dx$$

$$= \int_0^1 \left(1 + 2x\cos(1) - x^2 - 2x\cos(x^2) \right) dx$$

$$= \left[x + x^2\cos(1) - \frac{x^3}{3} - \sin(x^2) \right]_0^1$$

$$= 1 + \cos(1) - \frac{1}{3} - \sin(1)$$

$$= \boxed{\frac{2}{3} + \cos(1) - \sin(1)}$$

$\int y \sin(y^2)\, dy$

$u = y^2$

$\frac{1}{2} du = y\,dy$

$\frac{1}{2} \int \sin(u)\, du$

$= -\frac{\cos(u)}{2} + C$

$= -\frac{\cos(y^2)}{2} + C$

5. Consider the differential form $yx^2\,dx - x^2\,dy$.

(a) (3 points) Is this differential form exact?

$$\frac{\partial P}{\partial y} = x^2$$

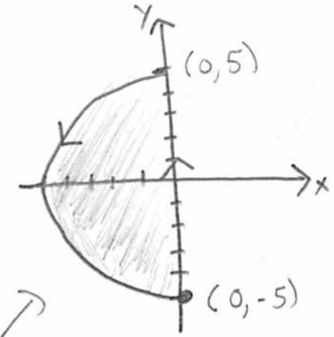
$$\frac{\partial Q}{\partial x}\ -2x$$

Not Exact!

(b) (6) Use *Green's Theorem* to evaluate the line integral

$$\int_C yx^2\,dx - x^2\,dy,$$

where C is the boundary of the semicircular region to the left of the y-axis and bounded by the circle $x^2 + y^2 = 25$.



$$\oint_C yx^2 dx - x^2 dy$$

$$= \iint_R (-2x - x^2)\, dA$$

Polar

$0 \le r \le 5$

$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

$$= -\int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \int_0^5 \left(2r\cos\theta + r^2\cos^2\theta\right) r\, dr\, d\theta$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^5 \left[2r^2\cos\theta + \frac{r^3}{2}\left(1+\cos(2\theta)\right)\right] dr\, d\theta$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\frac{250}{3}\cos\theta + \frac{625}{8}\left(1+\cos(2\theta)\right)\right] d\theta$$

$$= -\left[\frac{250}{3}\sin\theta + \frac{625}{8}\theta + \frac{625}{16}\sin(2\theta)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= -\left[-\frac{250}{3} + \frac{625}{8}\cdot\frac{3\pi}{2} - \frac{250}{3} - \frac{625}{8}\cdot\frac{\pi}{2}\right] = \boxed{\frac{500}{3} - \frac{625}{8}\pi}$$

(c) (6 points) Evaluate the integral from part (b) by evaluating the line integral(s) directly.

$C_1 =$ Semicircle: $x = 5\cos t$, $y = 5\sin t$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$ $\Rightarrow \vec{r}_1(t) = 5\cos t\,\vec{i} + 5\sin t\,\vec{j}$

$\vec{v}_1(t) = -5\sin t\,\vec{i} + 5\cos t\,\vec{j}$

$C_2 =$ line segment: $x = (1-t)(0) + t(0) = 0$

$y = (1-t)(-5) + t(5) = 10t - 5$ $\qquad 0 \le t \le 1$

$\vec{r}_2(t) = (10t-5)\vec{j}$, $\vec{v}_2(t) = 10\vec{j}$,

$$\oint_C yx^2\,dx - x^2\,dy = \int_{C_1} yx^2\,dx - x^2\,dy + \int_{C_2} yx^2\,dx - x^2\,dy$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(-625\cos^2 t\sin^2 t - 125\cos^3 t\right)dt + \int_0^1 0$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(-\frac{625}{4}\left(1-\cos^2(2t)\right) - 125\left(1-\sin^2 t\right)\cos t\right)dt$$

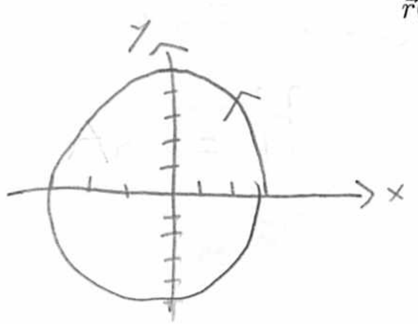
$$= -125\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{5}{4}\left(\frac{1}{2} - \frac{\cos(4t)}{2}\right) + \underbrace{(1-\sin^2 t)\cos t}_{(1-u^2)\,du}\right)dt$$

$$= -125\left[\frac{5}{8}t - \frac{5\sin(4t)}{32} + \sin t - \frac{\sin^3 t}{3}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= -125\left[\frac{15\pi}{16} - 1 + \frac{1}{3} - \frac{5\pi}{16} - 1 + \frac{1}{3}\right] = -125\left[\frac{10\pi}{16} - \frac{4}{3}\right] = \boxed{\frac{500}{3} - \frac{625}{8}\pi}$$

6. Use *Green's Theorem* to compute the area of the region R bounded by the ellipse C given by

$$\vec{r}(t) = 3\cos t\vec{i} + 5\sin t\vec{j}, \qquad 0 \le t \le 2\pi.$$



$$\text{Area} = \iint_R 1 \cdot dA$$

$$= \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} \left[3\cos t \cdot 5\cos t - 5\sin t(-3\sin t)\right] dt$$

$$= \frac{1}{2}\int_0^{2\pi} (15\cos^2 t + 15\sin^2 t)\, dt$$

$$= \frac{1}{2}(15 \cdot 2\pi)$$

$$= \boxed{15\pi}$$