Semidistributive Semilattices

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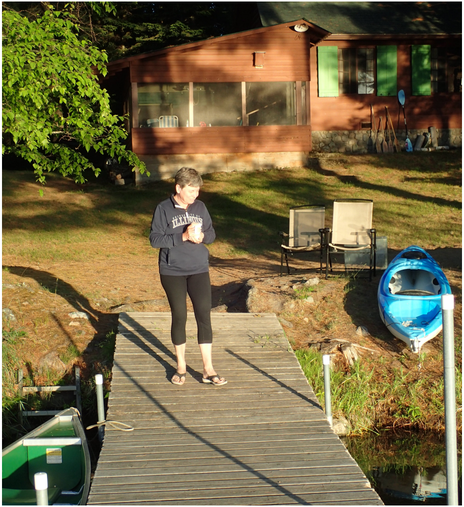
You're sitting there, minding your own business, when someone walks up and asks, "Does there exist a lattice with the following properties ..." **What do you do?**



Join semilattices with 0

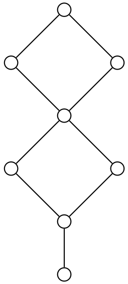
Think of a finite lattice as being given by a join semilattice presentation: $L = \langle X^{\vee} \mid \mathcal{R} \rangle$ where

- the elements of X are join irreducible,
- $\mathcal{R}$ is a collection of inclusions $p \leq r$ and (minimal) nontrivial join covers $p \leq \bigvee Q$.

This is equivalent to a closure system (or closure operator) description, so it should be familiar.

Test question

Is there a convex geometry whose congruence lattice is



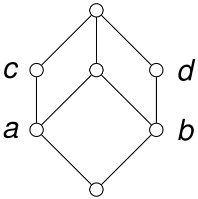
Translation: Does

$$L_1 = \langle \{a, b, c, d, e\}^{\vee} \mid a \leq e, \quad a \leq b \vee c, \quad b, c \leq d \vee e \rangle$$

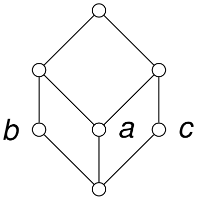
have unique irredundant join decompositions?

Three more examples

$$L_2 = \langle \{a, b, c, d\}^\vee \mid a \le c, \quad b \le d, \quad c \le a \vee d, \quad d \le b \vee c \rangle$$

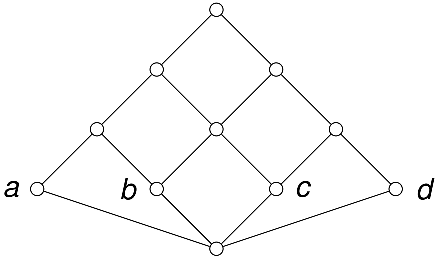


$$L_3 = \langle \{a, b, c\}^\vee \mid a \le b \vee c \rangle$$



Three more examples

$$L_4 = \langle \{a, b, c, d\}^\vee \mid b \leq a \vee c, \quad c \leq b \vee d, \quad b, c \leq a \vee d \rangle$$



The **join semidistributive law** (JSD) is

$$a \vee b = a \vee c \;\rightarrow\; a \vee b = a \vee (b \wedge c)$$

Fact 1: L is JSD iff every element of L has a unique non-refinable join decomposition.

Meet semidistributivity (MSD) is the dual.

Semidistributivity is both (SD= JSD + MSD).

Lower bounded lattices

For x, y join irreducible define $x \mathrel{\mathrm{D}} y$ if $y \in Y$ for some mntjc $x \leq \bigvee Y$.

L is **lower bounded** if it contains no D-cycle

$$x_0 \mathrel{\mathrm{D}} x_1 \mathrel{\mathrm{D}} x_2 \mathrel{\mathrm{D}} \ldots \mathrel{\mathrm{D}} x_{n-1} \mathrel{\mathrm{D}} x_0$$

Equivalently, L is lower bounded if the reflexive, transitive closure of the D relation is a partial order on $\mathrm{J}(L)$.

Fact 2: If L is lower bounded then it is JSD.

Testing LB

$$L_1 = \langle \{a, b, c, d, e\}^\vee \mid a \leq e, \quad a \leq b \vee c, \quad b, c \leq d \vee e \rangle$$

Lower bounded

$$L_2 = \langle \{a, b, c, d\}^\vee \mid a \leq c, \quad b \leq d, \quad \mathbf{c} \leq a \vee \mathbf{d}, \quad \mathbf{d} \leq b \vee \mathbf{c} \rangle$$

c D **d** D **c** so L_2 is not LB

$$L_3 = \langle \{a, b, c\}^\vee \mid a \leq b \vee c \rangle$$

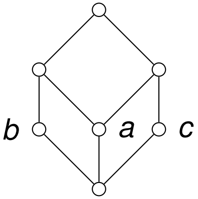
Lower bounded

$$L_4 = \langle \{a, b, c, d\}^\vee \mid \mathbf{b} \leq a \vee \mathbf{c}, \quad \mathbf{c} \leq \mathbf{b} \vee d, \quad b, c \leq a \vee d \rangle$$

b D **c** D **b** so L_4 is not LB (though it is JSD)

The K theorems

Assume $X = \mathrm{J}(L)$.
Define $x_\dagger = \bigvee\{y \in X : y < x\}$.
Define $K(x) = \{a \in L : a \text{ is maximal wrt } a \geq x_\dagger,\ a \not\geq x\}$.
Both $x_\dagger$ and $K(x)$ are easily computed.
Standard arguments give $\bigcup_{x \in X} K(x) = \mathrm{M}(L)$.



$$K(a) = \{b, c\}$$
$$K(b) = \{ac\}$$
$$K(c) = \{ab\}$$

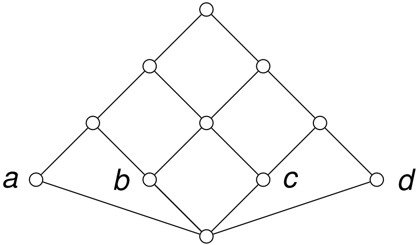
Fact 3: Let $L = X^{\vee}$ with $X = \mathrm{J}(L)$.

1. L is JSD iff $x \neq x'$ implies $K(x) \cap K(x') = \varnothing$.
2. L is MSD iff $|K(x)| = 1$ for all x.
3. L is SD iff K is a bijection twixt $\mathrm{J}(L)$ and $\mathrm{M}(L)$.

L_3 is JSD not MSD

Distributive lattices illustrate (3), the diamond M_3 is an example of **none of the above.**

L_4 is JSD not MSD

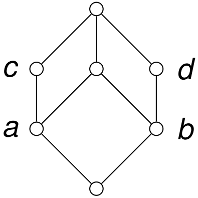


$$K(a) = \{bcd\}$$
$$K(b) = \{a, cd\}$$
$$K(c) = \{d, ab\}$$
$$K(d) = \{abc\}$$

Fact 4: In an MSD lattice, atoms are join prime.

The K theorems

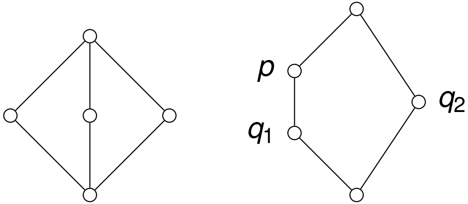
L_2 is MSD not JSD



$$K(a) = \{d\}$$
$$K(b) = \{c\}$$
$$K(c) = \{ab\}$$
$$K(d) = \{ab\}$$

Fact 5: L is a convex geometry iff every element of L has a unique irredundant join decomposition.

NON-EXAMPLES. M_3, the pentagon N.



Convex geometries

Fact 6: Let L be a finite JSD lattice. Then L is a convex geometry iff it has no mntjc $p \leq \bigvee Q$ with $q < p$ for some $q \in Q$.

EXAMPLE. L_1 is a convex geometry. (The element e is join prime.)

$$L_1 = \langle \{a, b, c, d, e\}^{\vee} \mid a \leq e, \quad a \leq b \vee c, \quad b, c \leq d \vee e \rangle$$

Corollary: An atomistic JSD lattice is a convex geometry.

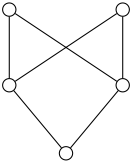
EXAMPLE. $\mathrm{Co}(P)$ is a convex geometry for any ordered set P

Back to the original question

Fact 7: Let D a finite distributive lattice with $D \cong \mathcal{O}(P)$, and let M denote the maximal members of P. TFAE:

1. $D \cong \operatorname{Con} L$ for a finite JSD lattice L.
2. $D \cong \operatorname{Con} G$ for a finite convex geometry G.
3. For all $x \in P \setminus M$, $|{\uparrow}x \cap M| \geq 2$.

EXAMPLE.



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An infinite, simple SD lattice

Fact 8: Every JSD lattice with 1 has **2** as a homomorphic image.

In particular, there is no finite, simple JSD lattice (or MSD lattice) except **2**. **Is there a simple semidistributive lattice besides 2?**

In order to adapt the methods discussed here, define a lattice L to be **strongly locally finite** if every interval $[u, v]$ is finite. Ralph and I found a presentation $\langle X, \mathcal{R} \rangle$ that is strongly locally finite, simple, SD.

Fact 9: There is an infinite, simple SD lattice.

Conclusion

You can understand some lattices that are too hard to draw!

Mahalo!