Distributive Subquasivariety Lattices

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In this dark, when we all talk at once, some of us must learn to whistle. - Walt Kelly

Varieties and quasivarieties

A **variety** is a class of algebras determined by equations, or more generally, a class of structures determined by atomic formulas.

$$x(yz) \approx (xy)z$$
$$x1 \approx x$$
$$x \leq y$$

A **quasivariety** is a class of algebras or structures determined by implications.

$$x^2 \approx 1 \to xy \approx yx$$
$$x \leq y \;\&\; y \leq z \to x \leq z$$
$$P(x) \to Q(x)$$

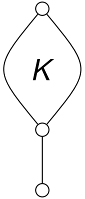
$\mathrm{L_v}(\mathcal{V})$ denotes the subvarieties contained in $\mathcal{V}$.

$\mathrm{L_q}(\mathcal{Q})$ denotes the subquasivarieties contained in $\mathcal{Q}$.

Subvariety lattices

Let $\mathsf{L_v}(\mathcal{V})$ denote the lattice of subvarieties of $\mathcal{V}$.

- $\mathsf{L_v}(\mathcal{V})$ is dually algebraic.
- If $\mathcal{V}$ is a variety of **algebras**, then $\mathsf{L_v}(\mathcal{V})$ satisfies Lampe's Zipper Condition:
  If $a_i \vee c = z$ for all $i \in I$ and $\bigwedge_{i \in I} a_i = 0$, then $c = z$.
- (Pigozzi and Tardos): If K is a dually algebraic lattice, then there is a variety $\mathcal{V}$ of algebras such that $1 + K \cong \mathsf{L_v}(V)$.
- For any dually algebraic lattice L, there is a **quasivariety $\mathcal{Q}$ of 1-element relational structures** such that $L \cong \mathsf{L_v}(\mathcal{Q})$.



$1 + K$

Subquasivariety lattices

Let $\mathrm{L_q}(\mathcal{K})$ denote the lattice of subquasivarieties of a quasivariety $\mathcal{K}$. In general, these lattices

- are dually algebraic,
- are atomic (every element is above an atom),
- are join semidistributive,

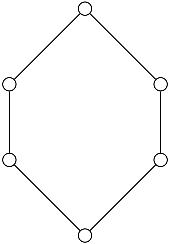
$$x \vee y \approx x \vee z \to x \vee y \approx x \vee (y \wedge z)$$

- support an equational closure operator,
- for many $\mathcal{K}$, satisfy no lattice identity.

Problem: Characterize subquasivariety lattices $\mathrm{L_q}(\mathcal{K})$?

Semidistributive but not $L_q(\mathcal{K})$

We will see many examples, **NOT** including these:



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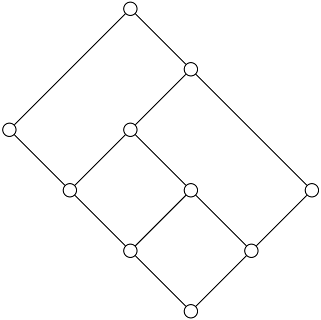

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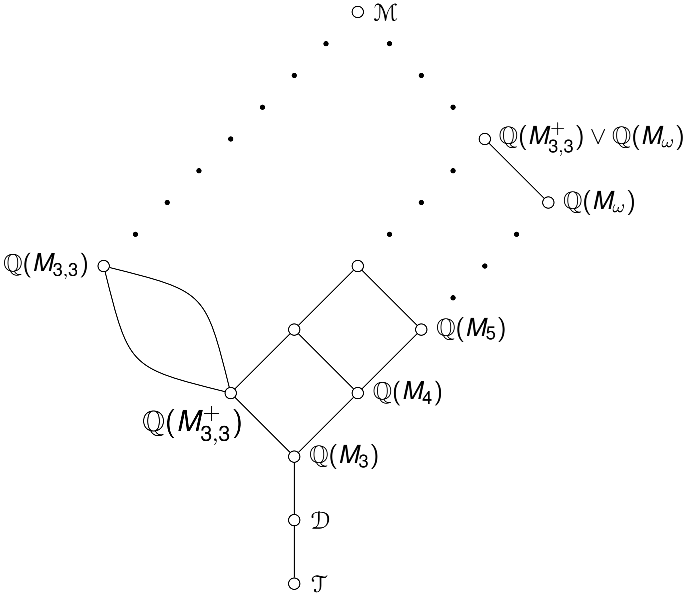
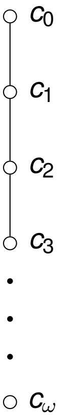
Quasivarieties of modular lattices (Grätzer and Lakser)



V. Tumanov (1983):
Every finite distributive lattice is isomorphic to $\mathbf{L_q}(\mathcal{K})$ for some quasivariety $\mathcal{K}$.

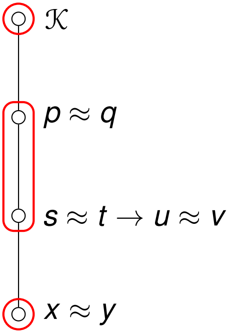
Which **infinite** distributive lattices are subquasivariety lattices?

First complication: $(\omega + 1)^d \ncong \mathrm{L_q}(\mathcal{K})$ because it is not atomic (though dually algebraic and distributive).

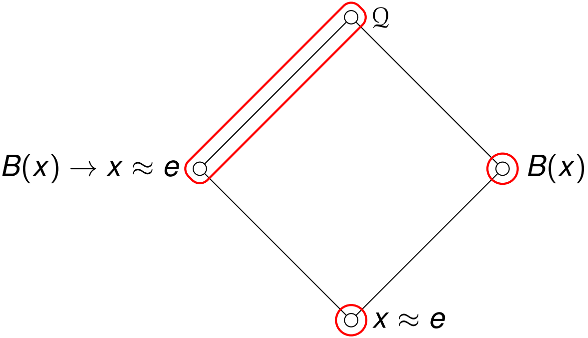


Second complication: subquasivariety lattices $\mathrm{L_q}(\mathcal{K})$ admit an equaclosure operator **(and most lattices don't!)**

For $\mathcal{S} \leq \mathcal{K}$, let $\Gamma(\mathcal{S}) = \mathbb{V}(\mathcal{S}) \cap \mathcal{K} = \mathbb{H}(\mathcal{S}) \cap \mathcal{K}$.

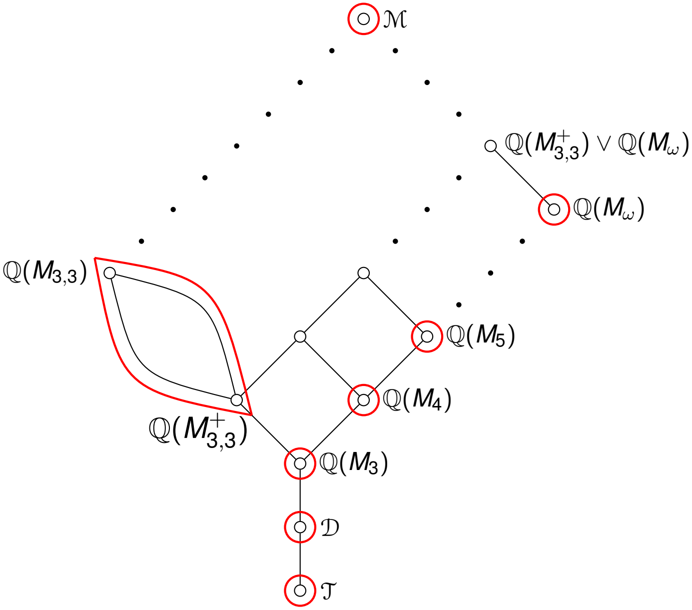


Another example.

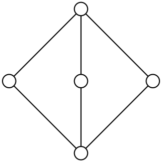


$\mathcal{Q}$ has a unary predicate B, a constant e, and the law $B(e)$.

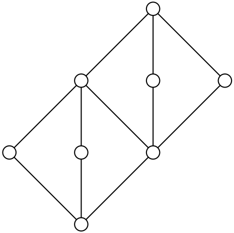
Equaclosure operator on $L_q(\mathcal{M})$



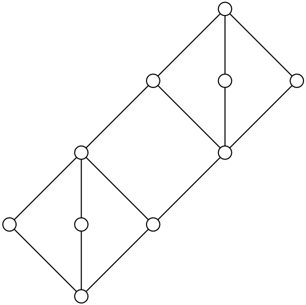
Small modular lattices



M_3

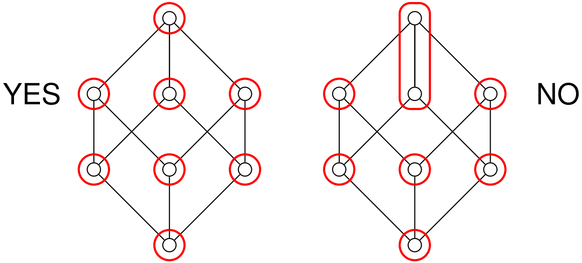


$M_{3,3}$



$M_{3,3}^{+}$

On a dually algebraic, distributive lattice with dually compact 0, the identity operator $\Gamma(x) = x$ is an equaclosure operator.



$\Gamma(x) = x$ means that every subquasivariety is a subvariety!

Classical lemma

TFAE for a complete lattice D.

- D is distributive, algebraic, and dually algebraic.
- D is distributive, dually algebraic, and upper continuous.
- $D \cong \mathcal{O}(P)$ for an ordered set P.

In that case, the least element 0_D is dually compact iff P has finitely many minimal elements, and every $p \in P$ is above at least one of them.

For a finite distributive lattice, $P \cong J(D) \cong M(D)$.

Distributive subquasivariety lattices

If the least element $\varnothing$ of $\mathcal{O}(P)$ is dually compact, then $\mathcal{O}(P)$ is isomorphic to $\mathsf{L_q}(\mathcal{Q})$ for some quasivariety $\mathcal{Q}$.

If D is a dually algebraic distributive lattice, then $1 + D$ is isomorphic to $\mathsf{L_q}(\mathcal{R})$ for some quasivariety $\mathcal{R}$.

There exist dually algebraic distributive lattices that are not isomorphic to any $\mathsf{L_q}(\mathcal{S})$ for $\mathcal{S}$ a quasivariety of structures in a language with equality, e.g., $(\omega + 1)^d$.

Every dually algebraic distributive lattice is isomorphic to $\mathsf{L_q}(\mathcal{U})$ for some quasivariety $\mathcal{U}$ in a language that does not contain equality.


ALOHA

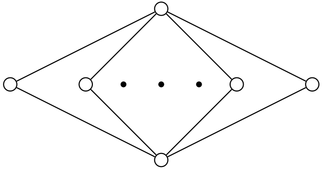
Primitive lattice varieties

A locally finite lattice variety $\mathcal{W}$ is said to be **primitive** if every subquasivariety $\mathcal{Q} \leq \mathcal{W}$ is a subvariety.

Equivalently, $\Gamma(\mathcal{Q}) = \mathcal{Q}$.

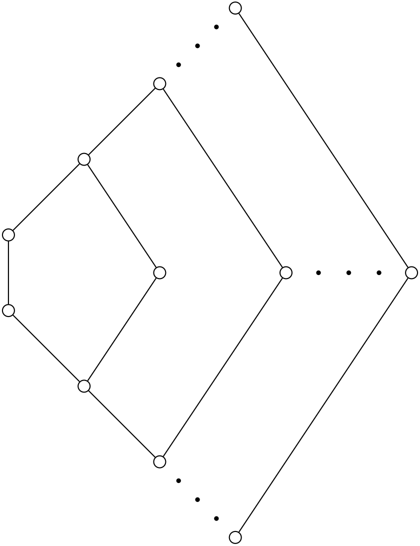
Example: the modular varieties $\mathbb{V}(M_k)$ for $1 \leq k \leq \omega$ are primitive.

If $\mathcal{W}$ is primitive, $\mathrm{L_q}(\mathcal{W}) = \mathrm{L_v}(\mathcal{W})$ is distributive.



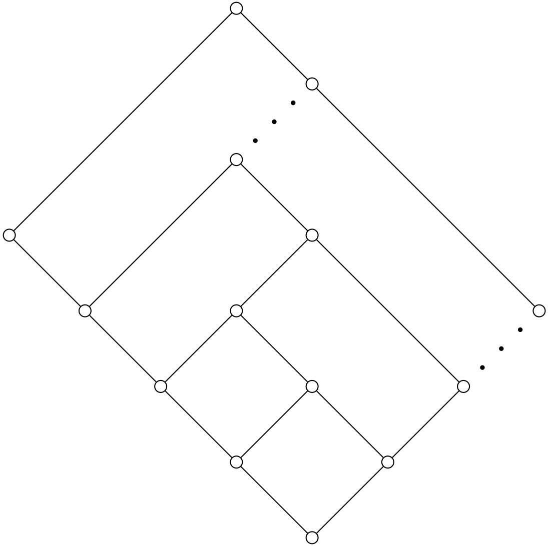
M_k

Lots more primitive lattice varieties



N_k

Lots more primitive lattice varieties



J_k

Even more ...

There is an uncountable ideal of primitive lattice varieties in the lattice Λ of all lattice varieties. (Jipsen, JBN)

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