MATH 216 WORKSHEET #6

For the following functions, find the gradient (in general), the gradient at the specified point, and the tangent (hyper-)plane.

(1) \( f(x, y) = x^4 + 2x^2y - y^2 \) at the point \((x_0, y_0) = (1, 2)\).

\[ \nabla f = \langle 4x^3 + 4y, 2x^2 - 2y \rangle \]
\[ \nabla f = \langle 12, -27 \rangle \text{ at } (1, 2) \]
\[ Z_0 = 1 + 4 - 4 = 1 \]
So the tangent plane is
\[ z - 1 = 12(x - 1) - 2(y - 2) \]
\[ z = 12x - 2y - 7 \]

(2) \( g(x, y) = 2e^x + y^2 \) at the point \((x_0, y_0) = (0, -2)\).

\[ \nabla g = \langle 2e^x, 2y \rangle \]
\[ \nabla g (0, -2) = \langle 2, -4 \rangle \]
\[ Z_0 = 2 + 4 = 6 \]
So the tangent plane is
\[ z - 6 = 2x - 4(y + 2) \]
\[ z = 2x - 4y - 2 \]

(3) \( h(x, y) = x + \frac{1}{y} + 2 \) at the point \((x_0, y_0) = (1, 4)\).

\[ \nabla h = \langle 1, -\frac{1}{y^2} \rangle \]
\[ \nabla h (1, 4) = \langle 1, -\frac{1}{16} \rangle \]
\[ Z_0 = \frac{13}{4} = \frac{13}{4} \]
So the tangent plane is
\[ z - \frac{13}{4} = 1(x - 1) - \frac{1}{16}(y - 4) \]
\[ z = x - \frac{1}{16} y + \frac{5}{2} \]
(4) \( f(x, y, z) = x^2 + yz + z^2 - 1 \) at the point \( (x_0, y_0, z_0) = (1, 0, 1) \).
\[
\nabla f = \begin{bmatrix} 2x \\ y+z \\ 2z \end{bmatrix}
\]
\[
\nabla f \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}
\]
\[
\mathbf{w} = \mathbf{e}_0 = \mathbf{i}
\]

Tangent hyperplane:
\[
\mathbf{w} - \mathbf{v} = 1(x-1) + 1(y-0) + 2(z-1)
\]
\[
\mathbf{w} = x+y+2z-2
\]

(5) \( g(x, y, z) = \sqrt{x^2+y^2+xyz} \) at the point \( (x_0, y_0, z_0) = (3, 4, 1) \).
\[
\nabla g = \frac{1}{2} \left( \frac{x}{\sqrt{x^2+y^2+xyz}} \right) \mathbf{e}_x + \frac{1}{2} \left( \frac{y}{\sqrt{x^2+y^2+xyz}} \right) \mathbf{e}_y + \frac{1}{2} \left( \frac{z}{\sqrt{x^2+y^2+xyz}} \right) \mathbf{e}_z
\]
\[
\nabla g \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 2 \end{bmatrix}
\]
\[
\mathbf{w} = 5 + 12 = 17
\]

Tangent hyperplane:
\[
\mathbf{w} - \mathbf{v} = \frac{3}{5}(x-3) + \frac{4}{5}(y-4) + 12(z-1)
\]

(6) \( m(x, y, z, t) = 2x - y + zt \) at the point \( (x_0, y_0, z_0, t_0) = (1, 2, 3, 4) \).
\[
\nabla m = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}
\]
\[
\nabla m \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 12 \end{bmatrix}
\]
\[
\mathbf{w} = 12
\]

Tangent hyperplane:
\[
\mathbf{w} - \mathbf{v} = 2(x-1) - 1(y-2) + 4(z-3) + 3(t-4)
\]
\[
\mathbf{w} = 2x-y+4z+3t-12
\]
(1) Find the directional derivative of \( z = x^2 + e^{xy} + y \) in the direction \( u = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \) at the point \( (x_0, y_0) = (2, 0) \).

\[
\nabla z = \langle 2x + ye^{xy}, xe^{xy} + 1 \rangle
\]

\[
\nabla z (2, 0) = \langle 2, 3 \rangle
\]

\[
D_u f = \langle 2, 3 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{5}{\sqrt{2}}
\]

(2) Find the directional derivative of \( z = \sin x + \frac{\tan x}{y + 1} \) in the direction \( u = \left( -\frac{3}{5}, \frac{4}{5} \right) \) at the point \( (x_0, y_0) = (0, 0) \).

\[
\nabla z = \langle \cos x + \sec^2 x / (y+1), -\frac{\tan x}{(y+1)^2} \rangle
\]

\[
\nabla z (0,0) = \langle 1, 0 \rangle
\]

\[
D_u f = \langle 1, 0 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle = -\frac{6}{5}
\]

(3) Find the directional derivative of \( w = x^2y - 2y^2 + xz^2 \) in the direction \( u = \left( \frac{5}{13}, 0, \frac{12}{13} \right) \) at the point \( (x_0, y_0, z_0) = (1, 1, 1) \).

\[
\nabla w = \langle 2xy + 2z^2, x^2 - 4y, x^2 \rangle
\]

\[
\nabla w (1,1,1) = \langle 4, -3, 1 \rangle
\]

\[
D_u f = \langle 4, -3, 1 \rangle \cdot \left\langle \frac{5}{13}, 0, \frac{12}{13} \right\rangle = \frac{20}{13} + 0 + \frac{12}{13} = \frac{32}{13}
\]
(4) For each of the functions in problems (1)–(3), find the second partial derivatives and put them into a Wronskian matrix.

\[ z = x^2 e^{xy} + y \]
\[ \nabla z = (2x y e^{xy}, x e^{xy} + 1) \]
\[ W = \begin{pmatrix}
 2 + y^2 e^{xy} & e^{xy} + x y e^{xy} \\
 2 + x e^{xy} & x^2 e^{xy}
\end{pmatrix} \]

\[ w = x^2 y - 2 y^2 + 2 x y \]
\[ \nabla w = (2 x y + 2 x y^2, x^2 - 4 y^2, x^2) \]
\[ W = \begin{pmatrix}
 2 y + 2 z & 2 x & 2 x \\
 2 x & -4 & 0 \\
 2 x & 0 & 0
\end{pmatrix} \]

(5) The potential energy of a (one-dimensional) spring is given by \( V = k x^2 \) where \( k \) is a constant and distance is measured from the equilibrium length of the spring. Find the force and write the differential equation governing its motion.

\[ F = m \frac{d^2 x}{dt^2} = -\frac{dv}{dx} = -2 k x \]

So the equation is

\[ m \frac{d^2 x}{dt^2} + 2 k x = 0 \]

(usually the potential for a spring is written as \( V = \frac{1}{2} k x^2 \).)

And the equation is

\[ m \frac{d^2 x}{dt^2} + k x = 0 \]

(6) The potential energy for a mass under the influence of gravity at the Earth’s surface is given by \( V = mgz \), where \( m \) denotes the mass (in kg) and \( g = 9.8 \text{m/sec}^2 \). Find the force and write the differential equation governing the motion.

\[ F = m \frac{d^2 z}{dt^2} = -\frac{dv}{dz} V = -mg \]

So the equation is

\[ \frac{d^2 z}{dt^2} = -g \]

(Cancelling the m’s).
If possible, find the function whose gradient is the given vector function.

1. \( f(x, y) = (x - 4xy, 5y - 2x^2) \)
   \[ \frac{\partial f}{\partial y} = -4x \quad \frac{\partial f}{\partial x} = -4x \]
   So it is exact.

   \[ V = \int P \, dx = \frac{x^2}{2} - 2x^2y + c(y) \]
   \[ V = \int Q \, dy = \frac{5y^2}{2} - 2x^2y + d(x) \]
   Combined, \( V = \frac{x^2}{2} - 2x^2y + \frac{5y^2}{2} + C \).

2. \( g(x, y) = (1 + \cos y - 14xy, -x \sin y - 7x^2) \)
   \[ \frac{\partial g}{\partial y} = -\sin y - 14x \]

   \[ V = \int P \, dx = x + x \cos y - 7x^2y + c(y) \]
   \[ V = \int Q \, dy = x \cos y - 7x^2y + d(x) \]
   So \( V = x + x \cos y - 7x^2y + C \).

3. \( h(x, y) = (x^2 + y + 1, e^y - x) \)
   \[ \frac{\partial h}{\partial y} = 1 \quad \frac{\partial h}{\partial x} = -1 \]
   So this is not a gradient.

4. \( k(x, y, z) = (1 + yz, 1 + xz, -1 + xy) \)
   \[ \frac{\partial k}{\partial y} = z = \frac{\partial k}{\partial x} \]
   \[ \frac{\partial k}{\partial z} = y = \frac{\partial k}{\partial x} \]
   \[ \frac{\partial k}{\partial y} = x = \frac{\partial k}{\partial y} \]
   \[ V = \int P \, dx = x + xy \]
   \[ V = \int Q \, dy = y + xy \]
   \[ V = \int R \, dz = -z + xy \]
   So combining
   \[ V = x + y - z + xy \]
   \[ V = x + y - z + xy + C \].
Solve these differential equations.

(5) \( \frac{dP}{dy} - 1 + 2y = 0 \)

\( \frac{dx}{dy} = \frac{dx}{x} \) so the equation is exact.

\( U = \int P \, dx = \frac{x^3}{3} + x + c(y) \)

\( V = \int Q \, dy = y^2 + d(x) \).

\( \text{Sol.} \quad \frac{x^3}{3} + x + y^2 = C \).

(6) \( 1 + y + (x + \cos y)y' = 0 \)

\( \frac{dx}{dy} = \frac{dx}{x} \)

\( U = \int 1 \, dx = x + xy + c(y) \)

\( V = \int x + \cos y \, dy = xy + \sin y + d(x) \).

\( \text{Sol.} \quad x + xy + \sin y = C \).

(7) \( 5 + ye^{xy} - y^3 + (xe^{xy} - 3xy^2)y' = 0 \)

\( \frac{dx}{dy} = 1 \cdot e^{xy} + ye^{xy} \frac{dx}{x} - 3y^2 = \frac{dx}{dx} = e^{xy} + xe^{xy} - 3y^2 \)

\( U = \int P \, dx = 5x + e^{xy} - xy^2 + c(y) \)

\( V = \int Q \, dy = e^{xy} - xy^2 + d(x) \).

\( \text{Sol.} \quad 5x + e^{xy} - xy^2 = C \).
MATH 216 REVIEW FOR SECOND EXAM

(1) Find the gradient and Wronskian matrix for the following functions.
(a) \( f(x, y) = x^2 y - \frac{x}{y} \)
\[ \nabla f = \left( \begin{array}{c} 2xy - \frac{1}{y} \\ x + \frac{x}{y^2} \end{array} \right) \]
\[ W = \begin{pmatrix} 2y & 2x + \frac{1}{y^2} \\ 2x + \frac{1}{y^2} & -\frac{2x}{y^3} \end{pmatrix} \]
(b) \( g(x, y, z) = \arctan x + 5xy^2 z^3 - 11y \)
\[ \nabla g = \left( \begin{array}{c} \frac{1}{1+x^2} + 5y^2 z^3 \\ 10xy z^3 - 11 \\ 15xy^2 z^2 \end{array} \right) \]
\[ W = \begin{pmatrix} -\frac{2x}{(1+x^2)^2} & 10yz^3 & 15y^2 z^2 \\ 10y z^3 & 10x z^3 & 30xy z^2 \\ 15y^2 z^2 & 30xy z^2 & 30y^2 z^2 \end{pmatrix} \]

(2) Find the tangent plane/hyperplane to the graph of the function at the given point.
(a) \( f(x, y) = x + \tan(xy) - y^2 \) at \((x_0, y_0) = (0, 2)\)
\[ \nabla f = \left( \begin{array}{c} \frac{1}{\cos^2(xy)} \\ x \sec^2(xy) \end{array} \right) \]
\[ \nabla f (0, 2) = \left( \begin{array}{c} 3 \\ -4 \end{array} \right) \]
\[ z = -4 \]
\[ \tan(p) \text{ normal: } z + 4 = 3x - 4(y - 2) \quad \text{or} \quad z = 3x - 4y + 4 \]
(b) \( g(x, y, z) = \ln(x + yz\sin y) - xz^2 \) at \((x_0, y_0, z_0) = (2, 0, -1)\)
\[ \nabla g = \left( \begin{array}{c} \frac{1}{x} - z^2 \\ xz \sin y + y \cos y \end{array} \right) \]
\[ \nabla g (2, 0, -1) = \left( \begin{array}{c} - \frac{1}{2} \\ 0 \\ 4 \end{array} \right) \]
\[ \mathbf{n}_0 = \left( \begin{array}{c} 0 \\ 0 \\ 4 \end{array} \right) \]
Tangent hyperplane,
\[ \mathbf{w} - \mathbf{n}_0 \cdot \mathbf{z} + \mathbf{z} = -\frac{1}{2} (x - 2) + 4(z + 1) \]
\[ \mathbf{w} = -\frac{1}{2} x + 4z + 3 + 4z. \]
(3) Find the directional derivative of ...
(a) the function \( f(x, y) = x^2e^{-y} \) at the point \((1, 0)\) in the direction of \( v = (1, -1) \).
\[
\nabla f = \left< 2xe^{-y}, -x^2e^{-y} \right>
\]
\[
\nabla f(1, 0) = \left< 2, -1 \right>
\]
\[
u = \left< \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right>
\]
\[
\nabla f \cdot \nu = \left< 2, -1 \right> \cdot \left< \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right> = \frac{3}{\sqrt{2}}
\]
(b) the function \( g(x, y, z) = x^2 + xy + e^{3z} \) at the point \((1, 2, 0)\) in the direction of \( v = (2, 2, 1) \).
\[
\nabla g = \left< 2x + y, x, 5e^{3z} \right>
\]
\[
\nabla g(1, 2, 0) = \left< 4, 1, 5 \right>
\]
\[
\nabla g \cdot \left< \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right> = \frac{8}{3} + \frac{2}{3} + \frac{5}{3} = 5.
\]

(4) You are standing on a hill. If you face north, the slope is 2 (it is a steep hill).
If you face east, the slope is -1.
(a) In what direction is the slope a maximum, and what is it?
(b) In what direction is the slope zero?
\[
\frac{df}{dy} = 2 \quad \text{and} \quad \frac{df}{dx} = -1 \quad \text{so} \quad \nabla f = \left< -1, 2 \right>.
\]
(a) \( \nabla f = \sqrt{5} \cdot \left< -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right> \) with \( \sqrt{5} \) being the magnitude and \( \left< -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right> \) being the direction.
(b) \( \nabla f \cdot \nu = 0 \) if \( \nu = \pm \left< \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right> \).

(5) Locate and classify the stationary points (if any) on the surfaces given below.
(a) \( z = 2x + 5y - 1 \)
\[
\nabla z = \left< 2, 5 \right> \quad \text{which is never zero.}
\]
No stationary point.

(b) \( z = x^2 - xy + y^2 - 2x + y \)
\[
\nabla z = (2x - y - 2, -x + 2y + 1)
\]
\[
\nabla z(2, 0) = \left< 0, 0 \right> \quad \text{if} \quad x = 1, \ y = 0.
\]
\[
\nabla z \left( \begin{array}{c} 2 \\ -1 \\ 2 \end{array} \right) \quad \text{and} \quad \det W = 3 \quad \text{so this is a minimum.}
\]

(c) \( z = x^3 + y^3 - 3xy \)
\[
\nabla z = (3x^2 - 3y, 3y^2 - 3x)
\]
\[
\nabla z = \left< 0, 0 \right> \quad \text{if} \quad \left< x, y \right> = \left< 1, 1 \right> \quad \text{or} \quad \left< x, y \right> = \left< 0, 0 \right>.
\]
\[
W = \left( \begin{array}{rr}
6x & -3 \\
-3 & 6y
\end{array} \right)
\]
\[
\begin{array}{c}
\text{minimum.} \\
\text{Saddle point.}
\end{array}
\]
(6) Find the function $V$ such that $\nabla V$ is the given vector function.

(a) $f(x, y) = (x + y^2, 2xy - 4)$

$$\frac{\partial p}{\partial y} = 2y = \frac{\partial q}{\partial x}$$

$$V_x = \int x + y^2 \, dx = \frac{x^2}{2} + xy^2 + C(y)$$

$$V_y = \int 2xy - y \, dy = xy^2 - \frac{y^2}{2} + d(x)$$

$$V_z = \frac{z}{2} + xy^2 - 4y + C.$$ 

(b) $g(x, y, z) = (1 + y^2z, y + 2xyz, y + z + xy^2)$

$$\frac{\partial p}{\partial x} = 2yz = \frac{\partial q}{\partial y}, \quad \frac{\partial q}{\partial z} = 2xy \neq 1 + 2xy = \frac{\partial p}{\partial x}$$

$$\frac{\partial p}{\partial y} = y^2 = \frac{\partial q}{\partial x}, \quad \text{so } \text{not a gradient}.$$ 

(7) Solve the differential equation.

(a) $y + 1 + (x - 1)y' = 0$

$$\frac{\partial p}{\partial x} = 0 = \frac{\partial q}{\partial y}$$

$$V_x = \int y + 1 + x \, dx = xy + x + C(y)$$

$$V_y = \int x - 1 \, dy = xy - y + d(x)$$

Solution: $xy + x - y = C.$

(b) $3x^2 + y^2 + 2xyy' = 0$

$$\frac{\partial p}{\partial y} = 2y = \frac{\partial q}{\partial x}$$

$$V_x = \int 3x^2 \, dx = \frac{3x^3}{2} + xy^2 + C(y)$$

$$V_y = \int 2y \, dy = xy^2 + d(x)$$

Solution: $x^3 + xy^2 = C.$