Appendix 1: Cardinals, Ordinals and Universal Algebra

In these notes we are assuming you have a working knowledge of cardinals and ordinals. Just in case, this appendix will give an informal summary of the most basic part of this theory. We also include an introduction to the terminology of universal algebra.

1. Ordinals

Let C be a well ordered set, i.e., a chain satisfying the descending chain condition (DCC). A *segment* of C is a proper ideal of C, which (because of the DCC) is necessarily of the form $\{c \in C : c < d\}$ for some $d \in C$.

Lemma. *Let C and D be well ordered sets. Then*

(1) *C is not isomorphic to any segment of itself.*
(2) *Either $C \cong D$, or C is isomorphic to a segment of D, or D is isomorphic to a segment of C.*

We say that two well ordered sets have the same *type* if $C \cong D$. An *ordinal* is an order type of well ordered sets. These are usually denoted by lower case Greek letters: α, β, γ, etc. For example, ω denotes the order type of the natural numbers, which is the smallest infinite ordinal. We can order ordinals by setting $\alpha \leq \beta$ if $\alpha \cong \beta$ or α is isomorphic to a segment of β. There are too many ordinals in the class of all ordinals to call this an ordered set without getting into set theoretic paradoxes, but we can say that locally it behaves like one big well ordered set.

Theorem. *Let β be an ordinal, and let B be the set of all ordinals α with $\alpha < \beta$, ordered by $\leq$. Then $B \cong \beta$.*

For example, ω is isomorphic to the collection of all finite ordinals.

Recall that the Zermelo well ordering principle (which is equivalent to the Axiom of Choice) says that every set can be well ordered. Another way of putting this is that every set can be indexed by ordinals,

$$X = \{x_\alpha : \alpha < \beta\}$$

for some β. Transfinite induction is a method of proof that involves indexing a set by ordinals, and then applying induction on the indices. This makes sense because the indices satisfy the DCC.

In doing transfinite induction, it is important to distinguish two types of ordinals. β is a *successor* ordinal if $\{\alpha : \alpha < \beta\}$ has a largest element. Otherwise, β is called a *limit* ordinal. For example, every finite ordinal is a successor ordinal, and ω is a limit ordinal.

2. Cardinals

We say that two sets X and Y have the same *cardinality*, written $|X| = |Y|$, if there exists a one-to-one onto map $f : X \to Y$. It is easy to see that "having the same cardinality" is an equivalence relation on the class of all sets, and the equivalence classes of this relation are called *cardinal numbers*. We will use lower case german letters such as $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{p}$ to denote unidentified cardinal numbers.

We order cardinal numbers as follows. Let X and Y be sets with $|X| = \mathfrak{m}$ and $|Y| = \mathfrak{n}$. Put $\mathfrak{m} \leq \mathfrak{n}$ if there exists a one-to-one map $f : X \rightarrowtail Y$ (equivalently, if there exists an onto map $g : Y \twoheadrightarrow X$). The Cantor-Bernstein theorem says that this relation is anti-symmetric: if $\mathfrak{m} \leq \mathfrak{n} \leq \mathfrak{m}$, then $\mathfrak{m} = \mathfrak{n}$, which is the hard part of showing that it is a partial order.

Theorem. *Let $\mathfrak{m}$ be any cardinal. Then there is a least ordinal α with $|\alpha| = \mathfrak{m}$.*

Theorem. *Any set of cardinal numbers is well ordered.*[1]

Now let $|X| = \mathfrak{m}$ and $|Y| = \mathfrak{n}$ with X and Y disjoint. We introduce operations on cardinals (which agree with the standard operations in the finite case) as follows.

$$\begin{aligned} \mathfrak{m} + \mathfrak{n} &= |X \cup Y| \\ \mathfrak{m} \cdot \mathfrak{n} &= |X \times Y| \\ \mathfrak{m}^{\mathfrak{n}} &= |\{f : Y \to X\}| \end{aligned}$$

It should be clear how to extend $+$ and $\cdot$ to arbitrary sums and products.

The basic arithmetic of infinite cardinals is fairly simple.

Theorem. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinals. Then*

(1) $\mathfrak{m} + \mathfrak{n} = \mathfrak{m} \cdot \mathfrak{n} = \max\{\mathfrak{m}, \mathfrak{n}\}$,
(2) $2^{\mathfrak{m}} > \mathfrak{m}$.

The finer points of the arithmetic can get complicated, but that will not bother us here. The following facts are used frequently.

Theorem. *Let X be an infinite set, $\mathcal{P}(X)$ the lattice of subsets of X, and $\mathcal{P}_f(X)$ the lattice of finite subsets of X. Then $|\mathcal{P}(X)| = 2^{|X|}$ and $|\mathcal{P}_f(X)| = |X|$.*

A fine little book [2] by Irving Kaplansky, *Set Theory and Metric Spaces*, is easy reading and contains the proofs of these theorems and more. The book *Introduction to Modern Set Theory* by Judith Roitman [4] is recommended for a slightly more advanced introduction.

[1] Again, there are too many cardinals to talk about the "set of all cardinals."

3. Universal Algebra

Once you have seen enough different kinds of algebras: vector spaces, groups, rings, semigroups, lattices, even semilattices, you should be driven to abstraction. The proper abstraction in this case is the general notion of an "algebra." Thus *universal algebra* is the study of the properties that different types of algebras have in common. Historically, lattice theory and universal algebra developed together, more like Siamese twins than cousins. In these notes we do not assume you know much universal algebra, but where appropriate we do use its terminology.

An *operation* on a set A is just a function $f : A^n \to A$ for some $n \in \omega$. An *algebra* is a system $\mathcal{A} = \langle A; \mathcal{F} \rangle$ where A is a nonempty set and $\mathcal{F}$ is a set of operations on A. Note that we allow infinitely many operations, but each has only finitely many arguments. For example, lattices have two binary operations, $\wedge$ and $\vee$. We use different fonts to distinguish between an algebra and the set of its elements, e.g., $\mathcal{A}$ and A.

Many algebras have distinguished elements, or constants. For example, groups have a unit element e, rings have both 0 and 1. Technically, these constants are nullary operations (with no arguments), and are included in the set $\mathcal{F}$ of operations. However, in these notes we commonly revert to a more old-fashioned notation and write them separately, as $\mathcal{A} = \langle A; \mathcal{F}, \mathcal{C} \rangle$, where $\mathcal{F}$ is the set of operations with at least one argument and $\mathcal{C}$ is the set of constants. There is no requirement that constants with different names, e.g., 0 and 1, be distinct.

A *subalgebra* of $\mathcal{A}$ is a subset S of A that is closed under the operations, i.e., if $s_1, \ldots, s_n \in S$ and $f \in \mathcal{F}$, then $f(s_1, \ldots, s_n) \in S$. This means in particular that all the constants of $\mathcal{A}$ are contained in S. If $\mathcal{A}$ has no constants, then we allow the empty set as a subalgebra (even though it is not properly an algebra). Thus the empty set is a sublattice of a lattice, but not a subgroup of a group. A nonempty subalgebra S of $\mathcal{A}$ can of course be regarded as an algebra $\mathcal{S}$ of the same type as $\mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are algebras with the same operation symbols (including constants), then a *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a mapping $h : A \to B$ that preserves the operations, i.e., $h(f(a_1, \ldots, a_n)) = f(h(a_1), \ldots, h(a_n))$ for all $a_1, \ldots, a_n \in A$ and $f \in \mathcal{F}$. This includes that $h(c) = c$ for all $c \in \mathcal{C}$.

A homomorphism that is one-to-one is called an *embedding*, and sometimes written $h : \mathcal{A} \rightarrowtail \mathcal{B}$ or $h : \mathcal{A} \leq \mathcal{B}$. A homomorphism that is both one-to-one and onto is called an *isomorphism*, denoted $h : \mathcal{A} \cong \mathcal{B}$.

These notions directly generalize notions that should be perfectly familiar to you for say groups or rings. Note that we have given only terminology, but no results. The basic theorems of universal algebra are included in the text, either in full generality, or for lattices in a form that is easy to generalize. For deeper results in universal algebra, there are several nice textbooks available, including *A Course in Universal Algebra* by S. Burris and H. P. Sankappanavar [1], and *Algebras, Lattices, Varieties* by R. McKenzie, G. McNulty and W. Taylor [3]. The former text [1] is out of print, but available for free downloading at Ralph Freese's website:

$$/www.math.hawaii.edu/\%7Eralph/Classes/619/.$$

Also on that website are other references, and universal algebra class notes by both Jarda Ježek and Kirby Baker.

References

1. S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York, 1980.
2. I. Kaplansky, *Set Theory and Metric Spaces*, Allyn and Bacon, Boston, 1972.
3. R. McKenzie, G. McNulty and W. Taylor, *Algebras, Lattices, Varieties*, vol. I, Wadsworth and Brooks-Cole, Belmont, CA, 1987.
4. J. Roitman, *Introduction to Modern Set Theory*, Wiley, New York, 1990.