OUTLINE OF CLOSURE OPERATORS

- A finite meet semilattice with 1 is a lattice.
- A *complete lattice* is an ordered set in which every subset has a meet and join.
- A *complete meet semilattice* is an ordered set in which every subset has a meet.
- Every complete meet semilattice is a complete lattice.
- Closure operators
  - $x \leq \gamma(x)$
  - $x \leq y \Rightarrow \gamma(x) \leq \gamma(y)$
  - $\gamma(\gamma(x)) = \gamma(x)$
- Closure systems: subsets of $\mathcal{P}(X)$ closed under arbitrary intersections (more generally, complete meets)
- Closure rules: implications $a \in S$ or $B \subseteq S \Rightarrow c \in S$
- The lattice of closed sets is a complete lattice.
- Every complete lattice can be so represented.
- Random examples from implications
- Topological closure
- Order ideals of an ordered set: $b \in I \Rightarrow a \in I$ over all pairs $a \leq b$ in P.
- Subgroups of a group, or more generally, subalgebras of an algebra
- Normal subgroups, ideals of a ring.
- Algebraic closure operators: from finitary rules. See Chapter 3 of the notes.
- Subspaces of a vector space: rules $\mathbf{0} \in S$ and $\mathbf{x}, \mathbf{y} \in S \Rightarrow c\mathbf{x} + d\mathbf{y} \in S$
- Flats of a geometry rules $\mathbf{x}, \mathbf{y} \in S \Rightarrow c\mathbf{x} + d\mathbf{y} \in S$ with $c + d = 1$
- The exchange property: $x \in \Gamma(\{y\} \cup Z)$ and $x \notin \Gamma(Z) \Rightarrow y \in \Gamma(\{x\} \cup Z)$
- Example: span in a vector space, flats of a geometry
- The convex hull of a set of points in $\mathbb{R}^n$: rules $\mathbf{x}, \mathbf{y} \in S \Rightarrow c\mathbf{x} + d\mathbf{y} \in S$ with $c + d = 1$ and $c, d \geq 0$
- The anti-exchange property: $\Gamma(Z) = Z$ and $x \in \Gamma(\{y\} \cup Z)$ and $x \notin \Gamma(Z) \Rightarrow y \notin \Gamma(\{x\} \cup Z)$
- Example: the convex hull operator

Date: September 19, 2014.

- Relative convex sets: restrict convex hull to fixed set T
- Convex subsets of an ordered set: $x, z \in S \Rightarrow y \in S$ for $x < y < z$ in P
- Galois connections: see Exercise 14 of Chapter 2.
- The MacNeille completion of an ordered set: apply the Galois connection closure to the relation $\leq$ contained in $P \times P$.
- Computing the closure: apply the closure rules recursively
- Proving the closure is what it is: to show that $\Gamma(S) = T$, show that $R \subseteq \Gamma(S)$ and that T is closed.
- Join as a closure operator on the nonzero join irreducibles of a finite lattice
- Bases for a finite lattice:
  1. All inclusions $p \leq q$ and $s \leq \bigvee T$
  2. Canonical direct basis: $p \leq q$ and $s \leq \bigvee T$ with T minimal w.r.t. set containment
  3. D-basis: $p \leq q$ and $s \leq \bigvee T$ with T minimal w.r.t. refinement
  4. GD basis
- The lattice of closure operators on a set