OUTLINE OF CLOSURE OPERATORS

• A finite meet semilattice with 1 is a lattice.
• A complete lattice is an ordered set in which every subset has a meet and join.
• A complete meet semilattice is an ordered set in which every subset has a meet.
• Every complete meet semilattice is a complete lattice.
• Closure operators
  \[ x \leq \gamma(x) \]
  \[ x \leq y \implies \gamma(x) \leq \gamma(y) \]
  \[ \gamma(\gamma(x)) = \gamma(x) \]
• Closure systems: subsets of \( P(X) \) closed under arbitrary intersections (more generally, complete meets)
• Closure rules: implications \( a \in S \) or \( B \subseteq S \implies c \in S \)
• The lattice of closed sets is a complete lattice.
• Every complete lattice can be so represented.
• Random examples from implications
• Topological closure
• Order ideals of an ordered set:
  \( b \in I \implies a \in I \) over all pairs \( a \leq b \) in \( P \).
• Subgroups of a group, or more generally, subalgebras of an algebra
• Normal subgroups, ideals of a ring.
• Algebraic closure operators: from finitary rules. See Chapter 3 of the notes.
• Subspaces of a vector space: rules \( 0 \in S \) and \( x, y \in S \implies cx + dy \in S \)
• Flats of a geometry rules \( x, y \in S \implies cx + dy \in S \) with \( c + d = 1 \)
• The exchange property: \( x \in \Gamma(\{y\} \cup Z) \) and \( x \notin \Gamma(Z) \implies y \in \Gamma(\{x\} \cup Z) \)
• Example: span in a vector space, flats of a geometry
• The convex hull of a set of points in \( \mathbb{R}^n \): rules \( x, y \in S \implies cx + dy \in S \) with \( c + d = 1 \) and \( c, d \geq 0 \)
• The anti-exchange property: \( \Gamma(Z) = Z \) and \( x \in \Gamma(\{y\} \cup Z) \) and \( x \notin \Gamma(Z) \implies y \notin \Gamma(\{x\} \cup Z) \)
• Example: the convex hull operator

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- Relative convex sets: restrict convex hull to fixed set $T$
- Convex subsets of an ordered set: $x, z \in S \Rightarrow y \in S$ for $x < y < z$ in $P$
- Galois connections: see Exercise 14 of Chapter 2.
- The MacNeille completion of an ordered set: apply the Galois connection closure to the relation $\leq$ contained in $P \times P$.
- Computing the closure: apply the closure rules recursively
- Proving the closure is what it is: to show that $\Gamma(S) = T$, show that $R \subseteq \Gamma(S)$ and that $T$ is closed.
- Join as a closure operator on the nonzero join irreducibles of a finite lattice
- Bases for a finite lattice:
  1. All inclusions $p \leq q$ and $s \leq \bigvee T$
  2. Canonical direct basis: $p \leq q$ and $s \leq \bigvee T$ with $T$ minimal w.r.t. set containment
  3. D-basis: $p \leq q$ and $s \leq \bigvee T$ with $T$ minimal w.r.t. refinement
  4. GD basis
- The lattice of closure operators on a set