Introduction

In the early 1890's, Richard Dedekind was working on a revised and enlarged edition of Dirichlet's *Vorlesungen über Zahlentheorie*, and asked himself the following question: Given three subgroups $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of an abelian group $\mathcal{G}$, how many different subgroups can you get by taking intersections and sums, e.g., $\mathcal{A}+\mathcal{B}$, $(\mathcal{A}+\mathcal{B})\cap\mathcal{C}$, etc. The answer, as we shall see, is 28 (Chapter 7). In looking at this and related questions, Dedekind was led to develop the basic theory of lattices, which he called *Dualgruppen*. His two papers on the subject, *Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler* (1897) and *Über die von drei Moduln erzeugte Dualgruppe* (1900), are classics, remarkably modern in spirit, which have inspired many later mathematicians.

"There is nothing new under the sun," and so Dedekind found. Lattices, especially distributive lattices and Boolean algebras, arise naturally in logic, and thus some of the elementary theory of lattices had been worked out earlier by Ernst Schröder in his book *Die Algebra der Logik*. Nonetheless, it is the connection between modern algebra and lattice theory, which Dedekind recognized, that provided the impetus for the development of lattice theory as a subject, and which remains our primary interest.

Unfortunately, Dedekind was ahead of his time in making this connection, and so nothing much happened in lattice theory for the next thirty years. Then, with the development of universal algebra in the 1930's by Garrett Birkhoff, Oystein Ore and others, Dedekind's work on lattices was rediscovered. From that time on, lattice theory has been an active and growing subject, in terms of both its application to algebra and its own intrinsic questions.

These notes are intended as the basis for a one-semester introduction to lattice theory. Only a basic knowledge of modern algebra is presumed, and I have made no attempt to be comprehensive on any aspect of lattice theory. Rather, the intention is to provide a textbook covering what we lattice theorists would like to think every mathematician should know about the subject, with some extra topics thrown in for flavor, all done thoroughly enough to provide a basis for a second course for the student who wants to go on in lattice theory or universal algebra.

It is a pleasure to acknowledge the contributions of students and colleagues to these notes. I am particularly indebted to Michael Tischendorf, Alex Pogel and the referee for their comments. *Mahalo* to you all.

Finally, I hope these notes will convey some of the beauty of lattice theory as I learned it from two wonderful teachers, Bjarni Jónsson and Bob Dilworth.