10. Finite Lattices and their Congruence Lattices

If memories are all I sing
I'd rather drive a truck.
–Ricky Nelson

In this chapter we want to study the structure of finite lattices, and how it is reflected in their congruence lattices. There are different ways of looking at lattices, each with its own advantages. For the purpose of studying congruences, it is useful to represent a finite lattice as the lattice of closed sets of a closure operator on its set of join irreducible elements. This is an efficient way to encode the structure, and will serve us well.[1]

The approach to congruences taken in this chapter is not the traditional one. It evolved from techniques developed over a period of time by Ralph McKenzie, Bjarni Jónsson, Alan Day, Ralph Freese and J. B. Nation for dealing with various specific questions (see [1], [4], [6], [7], [8], [9]). Gradually, the general usefulness of these methods dawned on us.

In the simplest case, recall that a finite distributive lattice $\mathcal{L}$ is isomorphic to the lattice of order ideals $\mathcal{O}(J(\mathcal{L}))$, where $J(\mathcal{L})$ is the ordered set of nonzero join irreducible elements of $\mathcal{L}$. This reflects the fact that join irreducible elements in a distributive lattice are join prime. In a nondistributive lattice, we seek a modification that will keep track of the ways in which one join irreducible is below the join of others. In order to do this, we must first develop some terminology.

Rather than just considering finite lattices, we can include with modest additional effort a larger class of lattices satisfying a strong finiteness condition. Recall that a lattice $\mathcal{L}$ is *principally chain finite* if no principal ideal of $\mathcal{L}$ contains an infinite chain (equivalently, every principal ideal $\downarrow x$ satisfies the ACC and DCC). In Theorem 11.1, we will see where this class arises naturally in an important setting.[2]

Recall that if $X, Y \subseteq L$, we say that X *refines* Y (written $X \ll Y$) if for each $x \in X$ there exists $y \in Y$ with $x \leq y$. It is easy to see that the relation $\ll$ is a quasiorder (reflexive and transitive), but not in general antisymmetric. Note $X \subseteq Y$ implies $X \ll Y$.

If $q \in J(\mathcal{L})$ is completely join irreducible, let q_* denote the unique element of L with $q \succ q_*$. Note that if $\mathcal{L}$ is principally chain finite, then q_* exists for each $q \in J(\mathcal{L})$.

[1] For an alternate approach, see Appendix. 3

[2] Many of the results in this chapter can be generalized to arbitrary lattices. However, these generalizations have not yet proved to be very useful unless one assumes at least the DCC.

A *join expression* of $a \in L$ is a finite set B such that $a = \bigvee B$. A join expression $a = \bigvee B$ is *minimal* if it is irredundant and B cannot be properly refined, i.e., $B \subseteq J(\mathcal{L})$ and $c \vee \bigvee(B - \{b\}) < a$ whenever $c < b \in B$. An equivalent way to write this technically is that $a = \bigvee B$ minimally if $a = \bigvee C$ and $C \ll B$ implies $B \subseteq C$.

A *join cover* of $p \in L$ is a finite set A such that $p \leq \bigvee A$. A join cover A of p is *minimal* if $\bigvee A$ is irredundant and A cannot be properly refined to another join cover of p, i.e., $p \leq \bigvee B$ and $B \ll A$ implies $A \subseteq B$.

We define a binary relation $\underline{D}$ on $J(\mathcal{L})$ as follows: $p \ \underline{D} \ q$ if there exists $x \in L$ such that $p \leq q \vee x$ but $p \not\leq q_* \vee x$. This relation will play an important role in our analysis of the congruences of a principally chain finite lattice.[3]

The following lemma summarizes some properties of principally chain finite lattices and the relation $\underline{D}$.

Lemma 10.1. *Let $\mathcal{L}$ be a principally chain finite lattice.*

(1) *If $b \not\leq a$ in $\mathcal{L}$, then there exists $p \in J(\mathcal{L})$ with $p \leq b$ and $p \not\leq a$.*
(2) *Every join expression in $\mathcal{L}$ refines to a minimal join expression, and every join cover refines to a minimal join cover.*
(3) *For $p, q \in J(\mathcal{L})$ we have $p \ \underline{D} \ q$ if and only if $q \in A$ for some minimal join cover A of p.*

Proof. (1) Since $b \not\leq a$ and $\downarrow b$ satisfies the DCC, the set $\{x \in \downarrow b : x \not\leq a\}$ has at least one minimal element p. Because $y < p$ implies $y \leq a$ for any $y \in L$, we have $\bigvee\{y \in L : y < p\} \leq p \wedge a < p$, and hence $p \in J(\mathcal{L})$ with $p_* = p \wedge a$.

(2) Suppose $\mathcal{L}$ contains an element s with a join representation $s = \bigvee F$ that does not refine to a minimal one. Since the DCC holds in $\downarrow s$, there is an element $t \leq s$ minimal with respect to having a join representation $t = \bigvee A$ which fails to refine to a minimal one. Clearly t is join reducible, and there is a proper, irredundant join expression $t = \bigvee B$ with $B \ll A$.

Let $B = \{b_1, \ldots, b_k\}$. Using the DCC on $\downarrow b_1$, we can find $c_1 \leq b_1$ such that $t = c_1 \vee b_2 \vee \ldots \vee b_k$, but c_1 cannot be replaced by any lower element: $t > u \vee b_2 \vee \ldots b_k$ whenever $u < c_1$. Now apply the same argument to b_2 and $\{c_1, b_2, \ldots, b_k\}$. After k such steps we obtain a join cover C that refines B and is minimal *pointwise*: no element can be replaced by a (single) lower element.

The elements of C may not be join irreducible, but each element of C is strictly below t, and hence has a minimal join expression. Choose a minimal join expression E_c for each $c \in C$. It is not hard to check that $E = \bigcup_{c \in C} E_c$ is a minimal join expression for t, and $E \ll C \ll B \ll A$, which contradicts the choice of t and B.

Now let $u \in L$ and let A be a join cover of u, i.e., $u \leq \bigvee A$. We can find $B \subseteq A$ such that $u \leq \bigvee B$ irredundantly. As above, refine B to a pointwise minimal

[3]Note that $\underline{D}$ is reflexive, i.e., $p \ \underline{D} \ p$ for all $p \in J(\mathbf{L})$. The relation D, defined similarly except that it requires $p \neq q$, is also important, and $\underline{D}$ stands for "D or equal to." For describing congruences, it makes more sense to use $\underline{D}$ rather than D.

join cover C. Now we know that minimal join expressions exist, so we may define $E = \bigcup_{c \in C} E_c$ exactly as before. Then E will be a minimal join cover of u, and again $E \ll C \ll B \ll A$.

(3) Assume $p \, \underline{D} \, q$, and let $x \in L$ be such that $p \leq q \vee x$ but $p \not\leq q_* \vee x$. By (2), we can find a minimal join cover A of p with $A \ll \{q, x\}$. Since $p \not\leq q_* \vee x$, we must have $q \in A$.

Conversely, if A is a minimal join cover of p, and $q \in A$, then we fulfill the definition of $p \, \underline{D} \, q$ by setting $x = \bigvee(A - \{q\})$. □

Now we want to define a closure operator on the join irreducible elements of a principally chain finite lattice. This closure operator should encode the structure of $\mathcal{L}$ in the same way the order ideal operator $\mathcal{O}$ does for a finite distributive lattice. For $S \subseteq J(\mathcal{L})$, let

$$\Gamma(S) = \{p \in J(\mathcal{L}) : p \leq \bigvee F \text{ for some finite } F \subseteq S\}.$$

It is easy to check that Γ is an algebraic closure operator. The compact (i.e., finitely generated) Γ-closed sets are of the form $\Gamma(F) = \{p \in J(\mathcal{L}) : p \leq \bigvee F\}$ for some finite subset F of $J(\mathcal{L})$. In general, we would expect these to be only a join subsemilattice of the lattice $\mathcal{C}_\Gamma$ of closed sets; however, for a principally chain finite lattice $\mathcal{L}$ the compact closed sets actually form an ideal (and hence a sublattice) of $\mathcal{C}_\Gamma$. For if $S \subseteq \Gamma(F)$ with F finite, then $S \subseteq \downarrow(\bigvee F)$, which satisfies the ACC. Hence $\{\bigvee G : G \subseteq S \text{ and } G \text{ is finite}\}$ has a largest element. So $\bigvee S = \bigvee G$ for some finite $G \subseteq S$, from which it follows that $\Gamma(S) = \Gamma(G)$, and $\Gamma(S)$ is compact. In particular, if $\mathcal{L}$ has a largest element 1, then every closed set will be compact.

With that preliminary observation out of the way, we proceed with our generalization of the order ideal representation for finite distributive lattices.

Theorem 10.2. *If $\mathcal{L}$ is a principally chain finite lattice, then the map ϕ with $\phi(x) = \{p \in J(\mathcal{L}) : p \leq x\}$ is an isomorphism of $\mathcal{L}$ onto the lattice of compact Γ-closed subsets of $J(\mathcal{L})$.*

Proof. Note that if $x = \bigvee A$ is a minimal join expression, then $\phi(x) = \Gamma(A)$, so $\phi(x)$ is indeed a compact Γ-closed set. The map ϕ is clearly order preserving, and it is one-to-one by part (1) of Lemma 10.1. Finally, ϕ is onto because $\Gamma(F) = \phi(\bigvee F)$ for each finite $F \subseteq J(\mathcal{L})$. □

To use this result, we need a structural characterization of Γ-closed sets.

Theorem 10.3. *Let $\mathcal{L}$ be a principally chain finite lattice. A subset C of $J(\mathcal{L})$ is Γ-closed if and only if*

(1) *C is an order ideal of $J(\mathcal{L})$, and*
(2) *if A is a minimal join cover of $p \in J(\mathcal{L})$ and $A \subseteq C$, then $p \in C$.*

Proof. It is easy to see that Γ-closed sets have these properties. Conversely, let $C \subseteq J(\mathcal{L})$ satisfy (1) and (2). We want to show $\Gamma(C) \subseteq C$. If $p \in \Gamma(C)$, then $p \le \bigvee F$ for some finite subset $F \subseteq C$. By Lemma 10.1(2), there is a minimal join cover A of p refining F; since C is an order ideal, $A \subseteq C$. But then the second closure property gives that $p \in C$, as desired. □

In words, Theorem 10.3 says that for principally chain finite lattices, Γ is determined by the order on $J(\mathcal{L})$ and the minimal join covers of elements of $J(\mathcal{L})$. Hence, by Theorem 10.2, $\mathcal{L}$ is determined by the same factors. Now we would like to see how much of this information we can extract from **Con** $\mathcal{L}$. The answer is, "not much." We will see that from **Con** $\mathcal{L}$ we can find $J(\mathcal{L})$ modulo a certain equivalence relation. We can determine nothing of the order on $J(\mathcal{L})$, nor can we recover the minimal join covers, but we can recover the $\underline{D}$ relation (up to the equivalence). This turns out to be enough to characterize the congruence lattices of principally chain finite lattices.

Now for a group $\mathcal{G}$, the map $\tau : \mathbf{Con}\, \mathcal{G} \to \mathcal{N}(\mathcal{G})$ given by $\tau(\theta) = \{x \in G : x\,\theta\,1\}$ is a lattice isomorphism. The next two theorems and corollary establish a similar correspondence for principally chain finite lattices.

Theorem 10.4. *Let $\mathcal{L}$ be a principally chain finite lattice. Let σ map* **Con** $\mathcal{L}$ *to the lattice of subsets $\mathcal{P}(J(\mathcal{L}))$ by*

$$\sigma(\theta) = \{p \in J(\mathcal{L}) : p\;\theta\;p_*\}.$$

Then σ is a one-to-one complete lattice homomorphism.

Proof. Clearly σ is order preserving: $\theta \le \psi$ implies $\sigma(\theta) \subseteq \sigma(\psi)$.

To see that σ is one-to-one, assume $\theta \not\le \psi$. Then there exists a pair of elements $a, b \in L$ with $a < b$ and $(a, b) \in \theta - \psi$. Since $(a, b) \notin \psi$, we also have $(x, b) \notin \psi$ for any element x with $x \le a$. Let $p \le b$ be minimal with respect to the property $p\,\psi\,x$ implies $x \not\le a$. We claim that p is join irreducible. If $y_1, \ldots, y_n < p$, then for each i there exists an x_i such that $y_i\,\psi\,x_i \le a$. Hence $\bigvee y_i\,\psi\,\bigvee x_i \le a$, so $\bigvee y_i < p$. Now $p = p \wedge b\,\theta\,p \wedge a \le p_*$, implying $p\,\theta\,p_*$, i.e., $p \in \sigma(\theta)$. But $(p, p_*) \notin \psi$ because $p_*\,\psi\,x \le a$ for some x; thus $p \notin \sigma(\psi)$. Therefore $\sigma(\theta) \not\subseteq \sigma(\psi)$.

It is easy to see that $\sigma(\bigwedge \theta_i) = \bigcap \sigma(\theta_i)$ for any collection of congruences θ_i $(i \in I)$. Since σ is order preserving, we have $\bigcup \sigma(\theta_i) \subseteq \sigma(\bigvee \theta_i)$, and it remains to show that $\sigma(\bigvee \theta_i) \subseteq \bigcup \sigma(\theta_i)$.

If $(p, p_*) \in \bigvee \theta_i$, then there exists a connecting sequence

$$p = x_0\,\theta_{i_1}\,x_1\,\theta_{i_2}\,x_2 \ldots x_{k-1}\,\theta_{i_k}\,x_k = p_*.$$

Let $y_j = (x_j \vee p_*) \wedge p$. Then $y_0 = p$, $y_k = p_*$, and $p_* \le y_j \le p$ implies $y_j \in \{p_*, p\}$ for each j. Moreover, we have $y_{j-1}\theta_{i_j}y_j$ for $j \ge 1$. There must exist a j with $y_{j-1} = p$ and $y_j = p_*$, whence $p\,\theta_{i_j}\,p_*$ and $p \in \sigma(\theta_{i_j}) \subseteq \bigcup \sigma(\theta_i)$. We conclude that σ also preserves arbitrary joins. □

Next we need to identify the range of σ.

Theorem 10.5. *Let $\mathcal{L}$ be a principally chain finite lattice, and let $S \subseteq J(\mathcal{L})$. Then $S = \sigma(\theta)$ for some $\theta \in$* **Con** *$\mathcal{L}$ if and only if $p \, \underline{D} \, q \in S$ implies $p \in S$.*

Proof. Let $S = \sigma(\theta)$. If $q \in S$ and $p \, \underline{D} \, q$, then $q \; \theta \; q_*$, and for some $x \in L$ we have $p \leq q \vee x$ but $p \nleq q_* \vee x$. Thus

$$p = p \wedge (q \vee x) \; \theta \; p \wedge (q_* \vee x) < p.$$

Hence $p \; \theta \; p_*$ and $p \in \sigma(\theta) = S$.

Conversely, assume we are given $S \subseteq J(\mathcal{L})$ satisfying the condition of the theorem. Then we must produce a congruence relation θ such that $\sigma(\theta) = S$. Let $T = J(\mathcal{L}) - S$, and note that T has the property that $q \in T$ whenever $p \, \underline{D} \, q$ and $p \in T$. Define

$$x \; \theta \; y \text{ if } \downarrow x \cap T = \downarrow y \cap T.$$

The motivation for this definition is outlined in the exercises: θ is the kernel of the *standard homomorphism* from $\mathcal{L}$ onto the join subsemilattice of $\mathcal{L}$ generated by $T \cup \{0\}$.

Three things should be clear: θ is an equivalence relation; $x \; \theta \; y$ implies $x \wedge z \; \theta \; y \wedge z$; and for $p \in J(\mathcal{L})$, $p \; \theta \; p_*$ if and only if $p \notin T$, i.e., $p \in S$. (The last statement will imply that $\sigma(\theta) = S$.) It remains to show that θ respects joins.

Assume $x \; \theta \; y$, and let $z \in L$. We want to show $\downarrow(x \vee z) \cap T \subseteq \downarrow(y \vee z) \cap T$, so let $p \in T$ and $p \leq x \vee z$. Then there exists a minimal join cover Q of p with $Q \ll \{x, z\}$. If $q \in Q$ and $q \leq z$, then of course $q \leq y \vee z$. Otherwise $q \leq x$, and since $p \in T$ and $p \, \underline{D} \, q$ (by Lemma 10.1(3)), we have $q \in T$. Thus $q \in \downarrow x \cap T = \downarrow y \cap T$, so $q \leq y \leq y \vee z$. It follows that $p \leq \bigvee Q \leq y \vee z$. This shows $\downarrow(x \vee z) \cap T \subseteq \downarrow(y \vee z) \cap T$; by symmetry, they are equal. Hence $x \vee z \; \theta \; y \vee z$. $\square$

In order to interpret the consequences of these two theorems, let $\trianglelefteq$ denote the transitive closure of $\underline{D}$ on $J(\mathcal{L})$. Then $\trianglelefteq$ is a quasiorder (reflexive and transitive), and so it induces an equivalence relation $\equiv$ on $J(\mathcal{L})$, modulo which $\trianglelefteq$ is a partial order, *viz.*, $p \equiv q$ if and only if $p \trianglelefteq q$ and $q \trianglelefteq p$. If we let $Q_{\mathcal{L}}$ denote the partially ordered set $(J(\mathcal{L})/\equiv, \trianglelefteq)$, then Theorem 10.5 translates as follows.

Corollary. *If $\mathcal{L}$ is a principally chain finite lattice, then* **Con** *$\mathcal{L} \cong \mathcal{O}(Q_{\mathcal{L}})$.*

Because the $\underline{D}$ relation is easy to determine, it is not hard to find $Q_{\mathcal{L}}$ for a finite lattice $\mathcal{L}$. Hence this result provides a reasonably efficient algorithm for determining the congruence lattice of a finite lattice. Hopefully, the exercises will convince you of this. As an application, we have the following characterization.

Corollary. *A principally chain finite lattice $\mathcal{L}$ is subdirectly irreducible if and only if $Q_{\mathcal{L}}$ has a least element.*

Now let us turn our attention to the problem of representing a given distributive algebraic lattice $\mathcal{D}$ as the congruence lattice of a lattice. Recall from Chapter 5 that

Fred Wehrung has shown that there are distributive algebraic lattices that are not isomorphic to the congruence lattice of any lattice. However, we can represent a large class that includes all finite lattices.

Not every distributive algebraic lattice is isomorphic to $\mathcal{O}(\mathcal{P})$ for an ordered set $\mathcal{P}$. Indeed, those that are so have a nice characterization.

Lemma 10.6. *The following are equivalent for a distributive algebraic lattice* $\mathcal{D}$.

(1) $\mathcal{D}$ *is isomorphic to the lattice of order ideals of an ordered set.*
(2) *Every element of* $\mathcal{D}$ *is a join of completely join prime elements.*
(3) *Every compact element of* $\mathcal{D}$ *is a join of (finitely many) join irreducible compact elements.*

Proof. An order ideal I is compact in $\mathcal{O}(\mathcal{P})$ if and only if it is finitely generated, i.e., $I = \downarrow p_1 \cup \dots \cup \downarrow p_k$ for some $p_1, \dots, p_k \in P$. Moreover, each $\downarrow p_i$ is join irreducible in $\mathcal{O}(\mathcal{P})$. Thus $\mathcal{O}(\mathcal{P})$ has the property (3).

Note that if $\mathcal{D}$ is a distributive algebraic lattice and p is a join irreducible compact element, then p is completely join prime. For if $p \leq \bigvee U$, then $p \leq \bigvee U'$ for some finite subset $U' \subseteq U$; as join irreducible elements are join prime in a distributive lattice, this implies $p \leq u$ for some $u \in U'$. On the other hand, a completely join prime element is clearly compact and join irreducible, so these elements coincide. If every compact element is a join of join irreducible compact elements, then so is every element of $\mathcal{D}$, whence (3) implies (2).

Now assume that the completely join prime elements of $\mathcal{D}$ are join dense, and let $\mathcal{P}$ denote the set of completely join prime elements with the order they inherit from $\mathcal{D}$. Then it is straightforward to show that the map $\phi : \mathcal{D} \to \mathcal{O}(\mathcal{P})$ given by $\phi(x) = \downarrow x \cap P$ is an isomorphism. □

Now it is not hard to find lattices where these conditions fail. Nonetheless, distributive algebraic lattices with the properties of Lemma 10.6 are a nice class (including all finite distributive lattices), and it behooves us to try to represent each of them as **Con** $\mathcal{L}$ for some principally chain finite lattice $\mathcal{L}$. We need to begin by seeing how $Q_{\mathcal{L}}$ can be recovered from **Con** $\mathcal{L}$.

Theorem 10.7. *Let* $\mathcal{L}$ *be a principally chain finite lattice. A congruence relation* θ *is join irreducible and compact in* **Con** $\mathcal{L}$ *if and only if* $\theta = \operatorname{con}(p, p_*)$ *for some* $p \in J$. *Moreover, for* $p, q \in J$, *we have* $\operatorname{con}(q, q_*) \leq \operatorname{con}(p, p_*)$ *iff* $q \trianglelefteq p$.

Proof. We want to use the representation **Con** $\mathcal{L} \cong \mathcal{O}(Q_{\mathcal{L}})$. Note that if Q is a partially ordered set and I is an order ideal of Q, then $I = \bigcup_{x \in I} \downarrow x$, and, of course, set union is the join operation in $\mathcal{O}(Q)$. Hence join irreducible compact ideals are exactly those of the form $\downarrow x$ for some $x \in Q$.

Applying these remarks to our situation, using the isomorphism, join irreducible compact congruences are precisely those with $\sigma(\theta) = \{q \in J(\mathcal{L}) : q \trianglelefteq p\}$ for some $p \in J(\mathcal{L})$. Recalling that $p \in \sigma(\theta)$ if and only if $p \; \theta \; p_*$, and $\operatorname{con}(p, p_*)$ is the least congruence with $p \; \theta \; p_*$, the conclusions of the theorem follow. □

Theorem 10.8. *Let $\mathcal{D}$ be a distributive algebraic lattice that is isomorphic to $\mathcal{O}(\mathcal{P})$ for some ordered set $\mathcal{P}$. Then there is a principally chain finite lattice $\mathcal{L}$ such that $\mathcal{D} \cong$ **Con** $\mathcal{L}$.*

Proof. We must construct $\mathcal{L}$ with $Q_{\mathcal{L}} \cong \mathcal{P}$. In view of Theorem 10.3 we should try to describe $\mathcal{L}$ as the lattice of finitely generated closed sets of a closure operator on an ordered set J. Let P^0 and P^1 be two unordered copies of the base set P of $\mathcal{P}$, disjoint except on the maximal elements of $\mathcal{P}$. Thus $J = P^0 \cup P^1$ is an antichain, and $p^0 = p^1$ if and only if p is maximal in $\mathcal{P}$. Define a subset C of J to be *closed* if $\{p^j, q^k\} \subseteq C$ implies $p^i \in C$ whenever $p < q$ in $\mathcal{P}$ and $\{i, j\} = \{0, 1\}$. Our lattice $\mathcal{L}$ will consist of all finite closed subsets of J, ordered by set inclusion.

It should be clear that we have made the elements of J atoms of $\mathcal{L}$ and

$$p^i \leq p^j \vee q^k$$

whenever $p < q$ in $\mathcal{P}$. Thus $p^i \, \underline{D} \, q^k$ iff $p \leq q$. (This is where you want only one copy of each maximal element). It remains to check that $\mathcal{L}$ is indeed a principally chain finite lattice with $Q_{\mathcal{L}} \cong \mathcal{P}$, as desired. The crucial observation is that the closure of a finite set is finite. We will leave this verification to the reader. □

Theorem 10.8 is due to R. P. Dilworth in the 1940's, but his proof was never published. The construction given is from George Grätzer and E. T. Schmidt [5].

We close this section with a new look at a pair of classic results. A lattice is said to be *relatively complemented* if $a < x < b$ implies there exists y such that $x \wedge y = a$ and $x \vee y = b$.[4]

Theorem 10.9. *If $\mathcal{L}$ is a principally chain finite lattice which is either modular or relatively complemented, then the relation $\underline{D}$ is symmetric on $J(\mathcal{L})$, and hence* **Con** $\mathcal{L}$ *is a Boolean algebra.*

Proof. First assume $\mathcal{L}$ is modular, and let $p \, \underline{D} \, q$ with $p \leq q \vee x$ but $p \not\leq q_* \vee x$. Using modularity, we have

$$(q \wedge (p \vee x)) \vee x = (q \vee x) \wedge (p \vee x) \geq p,$$

so $q \leq p \vee x$. On the other hand, if $q \leq p_* \vee x$, we would have

$$p = p \wedge (q \vee x) \leq p \wedge (p_* \vee x) = p_* \vee (x \wedge p) = p_*,$$

a contradiction. Hence $q \not\leq p_* \vee x$, and $q \, \underline{D} \, p$.

Now assume $\mathcal{L}$ is relatively complemented and $p \, \underline{D} \, q$ as above. Observe that a join irreducible element in a relatively complemented lattice must be an atom.

[4]Thus a relatively complemented lattice with 0 and 1 is complemented, but otherwise it need not be.

Hence $p_* = q_* = 0$, and given x such that $p \leq q \vee x$, $p \nleq x$, we want to find y such that $q \leq p \vee y$ and $q \nleq y$. Since $x < p \vee x \leq q \vee x$, we can choose y to be a relative complement of $p \vee x$ in the interval $(q \vee x)/x$. Then $p \vee y = p \vee x \vee y \geq q$, and $q \nleq y$ for otherwise would imply $p \leq y \wedge (p \vee x) = x$, a contradiction. Thus $q \, \underline{D} \, p$.

Finally, if $\underline{D}$ is symmetric, then $Q_{\mathcal{L}}$ is an antichain, and thus $\mathcal{O}(Q_{\mathcal{L}})$ is isomorphic to the Boolean algebra $\mathcal{P}(Q_{\mathcal{L}})$. $\square$

A lattice is *simple* if $|L| > 1$ and $\mathcal{L}$ has no proper nontrivial congruence relations, i.e., **Con** $\mathcal{L} \cong \mathbf{2}$. Theorem 10.9 says that a subdirectly irreducible, modular or relatively complemented, principally chain finite lattice must be simple.

In the relatively complemented case we get even more. Let $\mathcal{L}_i$ $(i \in I)$ be a collection of lattices with 0. The *direct sum* $\sum \mathcal{L}_i$ is the sublattice of the direct product consisting of all elements that are only finitely non-zero. Combining Theorems 10.2 and 10.9, we obtain relatively easily a fine result of Dilworth [2].

Theorem 10.10. *A relatively complemented principally chain finite lattice is a direct sum of simple (relatively complemented principally chain finite) lattices.*

Proof. Let $\mathcal{L}$ be a relatively complemented principally chain finite lattice. Then every element of L is a finite join of join irreducible elements, every join irreducible element is an atom, and the $\underline{D}$ relation is symmetric, i.e., $p \, \underline{D} \, q$ implies $p \equiv q$. We can write $J(\mathcal{L})$ as a disjoint union of $\equiv$-classes, $J(\mathcal{L}) = \dot{\bigcup}_{i \in I} A_i$. Let

$$L_i = \{x \in L : x = \bigvee F \text{ for some finite } F \subseteq A_i\}.$$

We want to show that the L_i's are ideals (and hence sublattices) of $\mathcal{L}$, and that $\mathcal{L} \cong \sum_{i \in I} \mathcal{L}_i$.

The crucial technical detail is this: *if $p \in J(\mathcal{L})$, $F \subseteq J(\mathcal{L})$ is finite, and $p \leq \bigvee F$, then $p \equiv f$ for some $f \in F$.* For F can be refined to a minimal join cover G of p, and since join irreducible elements are atoms, we must have $G \subseteq F$. But $p \, \underline{D} \, g$ (and hence $p \equiv g$) for every $g \in G$.

Now we can show that each $\mathcal{L}_i$ is an ideal of $\mathcal{L}$. Suppose $y \leq x \in L_i$. Then $x = \bigvee F$ for some $F \subseteq A_i$, and $y = \bigvee H$ for some minimal join expression $H \subseteq J(\mathcal{L})$. By the preceding observation, $H \subseteq A_i$, and thus $y \in L_i$.

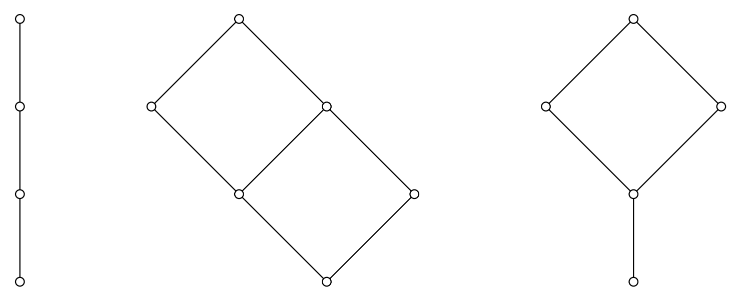
Define a map $\phi : \mathcal{L} \to \sum_{i \in I} \mathcal{L}_i$ by $\phi(x) = (x_i)_{i \in I}$, where $x_i = \bigvee {\downarrow} x \cap A_i)$. There are several things to check: that $\phi(x)$ is only finitely nonzero, that ϕ is one-to-one and onto, and that it preserves meets and joins. None is very hard, so we will only do the last one, and leave the rest to the reader.

We want to show that ϕ preserves joins, i.e., that $(x \vee y)_i = x_i \vee y_i$. It suffices to show that if $p \in J(\mathcal{L})$ and $p \leq (x \vee y)_i$, then $p \leq x_i \vee y_i$. Since $\mathcal{L}_i$ is an ideal, we have $p \in A_i$. Furthermore, since $p \leq x \vee y$, there is a minimal join cover F of p refining $\{x, y\}$. For each $f \in F$, we have $f \leq x$ or $f \leq y$, and $p \, \underline{D} \, f$ implies $f \in A_i$; hence $f \leq x_i$ or $f \leq y_i$. Thus $p \leq \bigvee F \leq x_i \vee y_i$. $\square$

Exercises for Chapter 10

1. Do Exercise 1 of Chapter 5 using the methods of this chapter.

2. Use the construction from the proof of Theorem 10.8 to represent the distributive lattices in Figure 10.1 as congruence lattices of lattices.



(a) (b) (c)

Figure 10.1

3. Let $a = \bigvee B$ be a join expression in a lattice $\mathcal{L}$. Prove that the following two properties (used to define minimality) really are equivalent.

(a) $B \subseteq J(\mathcal{L})$ and $c \vee \bigvee(B - \{b\}) < a$ whenever $c < b \in B$.

(b) $a = \bigvee C$ and $C \ll B$ implies $B \subseteq C$.

4. Let $\mathcal{P}$ be an ordered set satisfying the DCC, and let $\mathcal{Q}$ be the set of finite antichains of $\mathcal{P}$, ordered by $\ll$. Show that $\mathcal{Q}$ satisfies the DCC. (This argument is rather tricky, but it is the proper explanation of Lemma 10.1(2).)

5. Let p be a join irreducible element in a principally chain finite lattice. Show that p is join prime if and only if $p \, \underline{D} \, q$ implies $p = q$.

6. Let $\mathcal{L}$ be a principally chain finite lattice, and $p \in J(\mathcal{L})$. Prove that there is a congruence ψ_p on $\mathcal{L}$ such that, for all $\theta \in \mathbf{Con}\,\mathcal{L}$, $(p, p_*) \notin \theta$ if and only if $\theta \leq \psi_p$.

(More generally, the following is true: Given a lattice $\mathcal{L}$ and a filter F of $\mathcal{L}$, there is a unique congruence ψ_F maximal with respect to the property that $(x, f) \in \theta$ implies $x \in F$ for all $x \in L$ and $f \in F$.)

7. Prove that a distributive lattice is isomorphic to $\mathcal{O}(\mathcal{P})$ for some ordered set $\mathcal{P}$ if and only if it is algebraic and dually algebraic. (This extends Lemma 10.6.)

8. A complete lattice is *completely distributive* if it satisfies the identity

$$\bigwedge_{i \in I} \bigvee_{j \in J} x_{ij} = \bigvee_{f \in J^I} \bigwedge_{i \in I} x_{if(i)}$$

where J^I denotes the set of all $f : I \to J$.

(1) Show that this identity is equivalent to its dual.

(2) Prove that $\mathcal{O}(\mathcal{P}))$ is completely distributive for any ordered set $\mathcal{P}$.

(Indeed, a complete lattice $\mathcal{L}$ is completely distributive if and only if there is a complete surjective homomorphism $h : \mathcal{O}(\mathcal{P}) \to \mathcal{L}$, for some ordered set $\mathcal{P}$; see Raney [10].)

9. Let $\mathcal{L}$ be a principally chain finite lattice, and let $T \subseteq J(\mathcal{L})$ have the property that $p \,\underline{D}\, q$ and $p \in T$ implies $q \in T$.

(a) Show that the join subsemilattice $\mathcal{S}$ of $\mathcal{L}$ generated by $T \cup \{0\}$, i.e., the set of all $\bigvee F$ where F is a finite subset of $T \cup \{0\}$, is a lattice. ($\mathcal{S}$ need not a be sublattice of $\mathcal{L}$, because the meet operation is different.)

(b) Prove that the map $f : \mathcal{L} \twoheadrightarrow \mathcal{S}$ given by $f(x) = \bigvee {\downarrow} x \cap T)$ is a lattice homomorphism.

(c) Show that the kernel of f is the congruence relation θ in the proof of Theorem 10.5.

10. Prove that if $\mathcal{L}$ is a finite lattice, then $\mathcal{L}$ can be embedded into a finite lattice $\mathcal{K}$ such that **Con** $\mathcal{L} \cong$ **Con** $\mathcal{K}$ and every element of $\mathcal{K}$ is a join of atoms. (Michael Tischendorf)

11. Express the lattice of all finite subsets of a set X as a direct sum of two-element lattices.

12. Show that if $\mathcal{A}$ is a torsion abelian group, then the compact subgroups of $\mathcal{A}$ form a principally chain finite lattice (Khalib Benabdallah).

The main arguments in this chapter originated in a slightly different setting, geared towards application to lattice varieties [7], the structure of finitely generated free lattices [4], or finitely presented lattices [3]. The last three exercises give the version of these results which has proved most useful for these types of applications, with an example.

A lattice homomorphism $f : \mathcal{L} \to \mathcal{K}$ is *lower bounded* if for every $a \in K$, the set $\{x \in L : f(x) \geq a\}$ is either empty or has a least element, which is denoted $\beta(a)$. If f is onto, this is equivalent to saying that each congruence class of $\ker f$ has a least element. For example, if $\mathcal{L}$ satisfies the DCC, then every homomorphism $f : \mathcal{L} \to \mathcal{K}$ will be lower bounded. The dual condition is called *upper bounded*. These notions were introduced by Ralph McKenzie in [7].

13. Let $\mathcal{L}$ be a lattice with 0, $\mathcal{K}$ a finite lattice, and $f : \mathcal{L} \twoheadrightarrow \mathcal{K}$ a lower bounded, surjective homomorphism. Let $T = \{\beta(p) : p \in J(\mathcal{K})\}$. Show that:

(a) $T \subseteq J(\mathcal{L})$;

(b) $\mathcal{K}$ is isomorphic to the join subsemilattice $\mathcal{S}$ of $\mathcal{L}$ generated by $T \cup \{0\}$;

(c) for each $t \in T$, every join cover of t in $\mathcal{L}$ refines to a join cover of t contained in T.

14. Conversely, let $\mathcal{L}$ be a lattice with 0, and let T be a finite subset of $J(\mathcal{L})$ satisfying condition (c) of Exercise 13. Let $\mathcal{S}$ denote the join subsemilattice of $\mathcal{L}$ generated by $T \cup \{0\}$. Prove that the map $f : \mathcal{L} \twoheadrightarrow \mathcal{S}$ given by $f(x) = \bigvee {\downarrow} x \cap T)$ is a lower bounded lattice homomorphism with $\beta f(t) = t$ for all $t \in T$.

15. Let f be the (essentially unique) homomorphism from $FL(3)$ onto $\mathcal{N}_5$. Show that f is lower bounded. (By duality, f is also upper bounded.)

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