1. Ordered Sets

"And just how far would you like to go in?" he asked....
"Not too far but just far enough so's we can say that we've been there," said the first chief.
"All right," said Frank, "I'll see what I can do."
–Bob Dylan

In group theory, groups are defined algebraically as a model of permutations. The Cayley representation theorem then shows that this model is "correct": every group is isomorphic to a group of permutations. In the same way, we want to define a partial order to be an abstract model of set containment $\subseteq$, and then we should prove a representation theorem for partially ordered sets in terms of containment.

A *partially ordered set*, or more briefly just *ordered set*, is a system $\mathcal{P} = (P, \leq)$ where P is a nonempty set and $\leq$ is a binary relation on P satisfying, for all $x, y, z \in P$,

(1) $x \leq x$, (*reflexivity*)
(2) if $x \leq y$ and $y \leq x$, then $x = y$, (*antisymmetry*)
(3) if $x \leq y$ and $y \leq z$, then $x \leq z$. (*transitivity*)

The most natural example of an ordered set is $\mathfrak{P}(X)$, the collection of all subsets of a set X, ordered by $\subseteq$. Another familiar example is **Sub** $\mathcal{G}$, all subgroups of a group $\mathcal{G}$, again ordered by set containment. You can think of lots of examples of this type. Indeed, any nonempty collection Q of subsets of X, ordered by set containment, forms an ordered set.

More generally, if $\mathcal{P}$ is an ordered set and $Q \subseteq P$, then the restriction of $\leq$ to Q is a partial order, leading to a new ordered set $\mathcal{Q}$.

The set $\Re$ of real numbers with its natural order is an example of a rather special type of partially ordered set, namely a totally ordered set, or chain. $\mathcal{C}$ is a *chain* if for every $x, y \in C$, either $x \leq y$ or $y \leq x$. At the opposite extreme we have *antichains*, ordered sets in which $\leq$ coincides with the equality relation $=$.

We say that x *is covered by* y in $\mathcal{P}$, written $x \prec y$, if $x < y$ and there is no $z \in P$ with $x < z < y$. It is clear that the covering relation determines the partial order in a finite ordered set $\mathcal{P}$. In fact, the order $\leq$ is the smallest reflexive, transitive relation containing $\prec$. We can use this to define a *Hasse diagram* for a finite ordered set $\mathcal{P}$: the elements of P are represented by points in the plane, and a line is drawn from a up to b precisely when $a \prec b$. In fact this description is not precise, but it

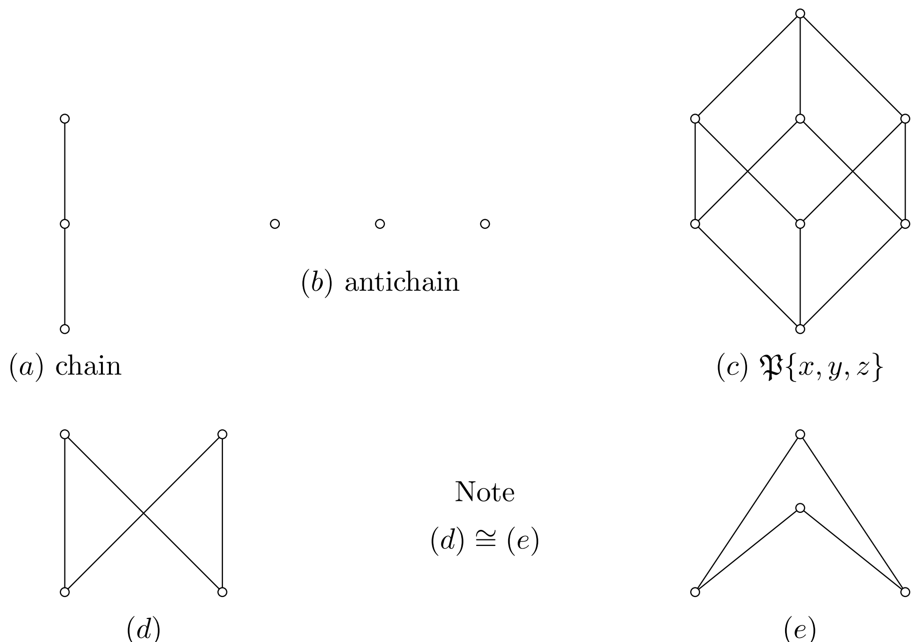

FIGURE 1.1

is close enough for government purposes. In particular, we can now generate lots of examples of ordered sets using Hasse diagrams, as in Figure 1.1.

The natural maps associated with the category of ordered sets are the *order preserving* maps, those satisfying the condition $x \leq y$ implies $f(x) \leq f(y)$. We say that $\mathcal{P}$ is *isomorphic* to $\mathcal{Q}$, written $\mathcal{P} \cong \mathcal{Q}$, if there is a map $f : P \to Q$ which is one-to-one, onto, and both f and f^{-1} are order preserving, i.e., $x \leq y$ iff $f(x) \leq f(y)$.

With that we can state the desired representation of any ordered set as a system of sets ordered by containment.

Theorem 1.1. *Let $\mathcal{Q}$ be an ordered set, and let $\phi : Q \to \mathfrak{P}(Q)$ be defined by*

$$\phi(x) = \{y \in Q : y \leq x\}.$$

Then Q is isomorphic to the range of ϕ ordered by $\subseteq$.

Proof. If $x \leq y$, then $z \leq x$ implies $z \leq y$ by transitivity, and hence $\phi(x) \subseteq \phi(y)$. Since $x \in \phi(x)$ by reflexivity, $\phi(x) \subseteq \phi(y)$ implies $x \leq y$. Thus $x \leq y$ iff $\phi(x) \subseteq \phi(y)$. That ϕ is one-to-one then follows by antisymmetry. □

A subset I of $\mathcal{P}$ is called an *order ideal* if $x \leq y \in I$ implies $x \in I$. The set of all order ideals of $\mathcal{P}$ forms an ordered set $\mathcal{O}(\mathcal{P})$ under set inclusion. The map

ϕ of Theorem 1.1 embeds $\mathcal{Q}$ in $\mathcal{O}(\mathcal{Q})$. Note that we have the additional property that the intersection of any collection of order ideals of $\mathcal{P}$ is again in an order ideal (which may be empty). Likewise, the union of a collection of order ideals is an order ideal.

Given an ordered set $\mathcal{P} = (P, \leq)$, we can form another ordered set $\mathcal{P}^d = (P, \leq^d)$, called the *dual* of $\mathcal{P}$, with the order relation defined by $x \leq^d y$ iff $y \leq x$. In the finite case, the Hasse diagram of $\mathcal{P}^d$ is obtained by simply turning the Hasse diagram of $\mathcal{P}$ upside down (see Figure 1.2). Many concepts concerning ordered sets come in dual pairs, where one version is obtained from the other by replacing "$\leq$" by "$\geq$" throughout.

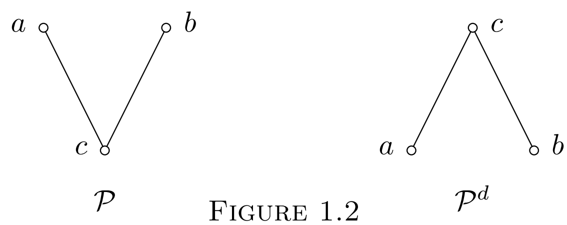

FIGURE 1.2

For example, a subset F of $\mathcal{P}$ is called an *order filter* if $x \geq y \in F$ implies $x \in F$. An order ideal of $\mathcal{P}$ is an order filter of $\mathcal{P}^d$, and *vice versa*.

An ideal or filter determined by a single element is said to be *principal*. We denote principal ideals and principal filters by

$$\downarrow x = \{y \in P : y \leq x\},$$
$$\uparrow x = \{y \in P : y \geq x\},$$

respectively.

The ordered set $\mathcal{P}$ has a *maximum* (or *greatest*) element if there exists $x \in P$ such that $y \leq x$ for all $y \in P$. An element $x \in P$ is *maximal* if there is no element $y \in P$ with $y > x$. Clearly these concepts are different. *Minimum* and *minimal* elements are defined dually.

The next lemma is simple but particularly important.

Lemma 1.2. *The following are equivalent for an ordered set $\mathcal{P}$.*

(1) *Every nonempty subset $S \subseteq P$ contains an element minimal in S.*

(2) *$\mathcal{P}$ contains no infinite descending chain*

$$a_0 > a_1 > a_2 > \dots$$

(3) *If*

$$a_0 \geq a_1 \geq a_2 \geq \dots$$

in $\mathcal{P}$, then there exists k such that $a_n = a_k$ for all $n \geq k$.

Proof. The equivalence of (2) and (3) is clear, and likewise that (1) implies (2). There is, however, a subtlety in the proof of (2) implies (1). Suppose $\mathcal{P}$ fails (1) and that $S \subseteq P$ has no minimal element. In order to find an infinite descending chain in S, rather than just arbitrarily long finite chains, we must use the Axiom of Choice. One way to do this is as follows.

Let f be a choice function on the subsets of S, i.e., f assigns to each nonempty subset $T \subseteq S$ an element $f(T) \in T$. Let $a_0 = f(S)$, and for each $i \in \omega$ define $a_{i+1} = f(\{s \in S : s < a_i\})$; the argument of f in this expression is nonempty because S has no minimal element. The sequence so defined is an infinite descending chain, and hence $\mathcal{P}$ fails (2). □

The conditions described by the preceding lemma are called the *descending chain condition* (DCC). The dual notion is called the *ascending chain condition* (ACC). These conditions should be familiar to you from ring theory (for ideals). The next lemma just states that ordered sets satisfying the DCC are those for which the principle of induction holds.

Lemma 1.3. *Let $\mathcal{P}$ be an ordered set satisfying the DCC. If $\varphi(x)$ is a statement such that*

(1) *$\varphi(x)$ holds for all minimal elements of P, and*
(2) *whenever $\varphi(y)$ holds for all $y < x$, then $\varphi(x)$ holds,*

then $\varphi(x)$ is true for every element of P.

Note that (1) is in fact a special case of (2). It is included in the statement of the lemma because in practice minimal elements usually require a separate argument (like the case $n = 0$ in ordinary induction).

The proof is immediate. The contrapositive of (2) states that the set $F = \{x \in P : \varphi(x) \text{ is false}\}$ has no minimal element. Since $\mathcal{P}$ satisfies the DCC, F must therefore be empty.

We now turn our attention more specifically to the structure of ordered sets. Define the *width* of an ordered set $\mathcal{P}$ by

$$w(\mathcal{P}) = sup\{|A| : A \text{ is an antichain in } \mathcal{P}\}$$

where $|A|$ denotes the cardinality of A.[1] A second invariant is the *chain covering number* $c(\mathcal{P})$, defined to be the least cardinal γ such that P is the union of γ chains in $\mathcal{P}$. Because no chain can contain more than one element of a given antichain, we must have $|A| \leq |I|$ whenever A is an antichain in $\mathcal{P}$ and $P = \bigcup_{i \in I} C_i$ is a chain covering. Therefore

$$w(\mathcal{P}) \leq c(\mathcal{P})$$

[1] Note that the width function $w(\mathcal{P})$ does not distinguish, for example, between ordered sets that contain arbitrarily large finite antichains and those that contain a countably infinite antichain. For this reason, in ordered sets of infinite width it is sometimes useful to consider the function $\mu(\mathcal{P})$, which is defined to be the least cardinal κ such that $\kappa + 1 > |A|$ for every antichain A of $\mathcal{P}$. We will restrict our attention to $w(\mathcal{P})$.

for any ordered set $\mathcal{P}$. The following result, due to R. P. Dilworth [2], says in particular that if $\mathcal{P}$ is finite, then $w(\mathcal{P}) = c(\mathcal{P})$.

Theorem 1.4. *If $w(\mathcal{P})$ is finite, then $w(\mathcal{P}) = c(\mathcal{P})$.*

Our discussion of the proof will take the scenic route. We begin with the case when $\mathcal{P}$ is finite, using Ralph Freese's modification of H. Tverberg's proof [17].

Proof in the finite case. We need to show $c(\mathcal{P}) \le w(\mathcal{P})$, which is done by induction on $|P|$. Let $w(\mathcal{P}) = k$, so that every maximal-sized antichain has k elements.

First, suppose there exists a k-element antichain $A = \{a_1, \dots, a_k\}$ that does not consist entirely of maximal elements of $\mathcal{P}$, or entirely of minimal elements. Set

$$L = \{x \in P : x \le a_i \text{ for some } i\},$$
$$U = \{x \in P : x \ge a_j \text{ for some } j\}.$$

Since every element of P is comparable with some element of A, we have $P = L \cup U$, while $A = L \cap U$. Moreover, since A contains at least one element that is not maximal in $\mathcal{P}$, we have $|L| < |P|$. Dually, $|U| < |P|$. Hence L is a union of k chains, $L = D_1 \cup \cdots \cup D_k$, and similarly $U = E_1 \cup \cdots \cup E_k$ as a union of chains. Each of these chains must contain exactly one a_i, so by renumbering (if necessary) we may assume that $a_i \in D_i \cap E_i$ for $1 \le i \le k$, so that $C_i = D_i \cup E_i$ is a chain. Thus

$$P = L \cup U = C_1 \cup \cdots \cup C_k$$

represents $\mathcal{P}$ as a union of k chains.

Hence we may assume that the only k-element antichains in $\mathcal{P}$ are its maximal elements, or the minimal elements, or both. Let $A = \{a_1, \dots, a_\ell\}$ be the maximal elements of $\mathcal{P}$, and let $B = \{b_1, \dots, b_m\}$ be the minimal elements. Both these are maximal antichains, and at least one of ℓ, m is k. Now $b_1 \le a_j$ for some j. Let $C_1 = \{b_1, a_j\}$, and note that $w(\mathcal{P} - C_1) = k - 1$. Thus $\mathcal{P} - C_1$ is a union of $k - 1$ chains, and the desired result follows. □

So now we want to consider an infinite ordered set $\mathcal{P}$ of finite width k. Not surprisingly, we will want to use one of the 210 equivalents of the Axiom of Choice! (See H. Rubin and J. Rubin [14].) This requires some standard terminology.

Let $\mathcal{P}$ be an ordered set, and let S be a subset of P. We say that an element $x \in P$ is an *upper bound* for S if $x \ge s$ for all $s \in S$. An upper bound x need not belong to S. We say that x is the *least upper bound* for S if x is an upper bound for S and $x \le y$ for every upper bound y of S. If the least upper bound of S exists, then it is unique. *Lower bound* and *greatest lower bound* are defined dually.

Theorem 1.5. *The following set theoretic axioms are equivalent.*

(1) (Axiom of Choice) *If X is a nonempty set, then there is a map $\phi : \mathfrak{P}(X) \to X$ such that $\phi(A) \in A$ for every nonempty $A \subseteq X$.*

(2) (ZERMELO WELL-ORDERING PRINCIPLE) *Every nonempty set admits a well-ordering (a total order satisfying the DCC).*
(3) (HAUSDORFF MAXIMALITY PRINCIPLE) *Every chain in an ordered set $\mathcal{P}$ can be embedded in a maximal chain.*
(4) (ZORN'S LEMMA) *If every chain in an ordered set $\mathcal{P}$ has an upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains a maximal element.*
(5) *If every chain in an ordered set $\mathcal{P}$ has a least upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains a maximal element.*

The proof of Theorem 1.5 is given in Appendix 2.

Our plan is to use Zorn's Lemma to prove the compactness theorem (due to K. Gödel [6]); then, following a suggestion of Bjarni Jónsson, use the compactness theorem to prove the infinite case of Dilworth's theorem. We need to first recall some of the basics of sentential logic.

Let S be a set, whose members will be called sentence symbols. Initially the sentence symbols carry no intrinsic meaning; in applications they will correspond to various mathematical statements.

We define *well formed formulas* (wff) on S by the following rules.

(1) Every sentence symbol is a wff.
(2) If α and β are wffs, then so are $(\neg\alpha)$, $(\alpha \text{ AND } \beta)$ and $(\alpha \text{ OR } \beta)$.
(3) Only symbols generated by the first two rules are wffs.

The set of all wffs on S is denoted by $\overline{S}$.[2]

A *truth assignment* on S is a map $\nu : S \to \{T, F\}$. Each truth assignment has a natural extension $\overline{\nu} : \overline{S} \to \{T, F\}$. The map $\overline{\nu}$ is defined recursively by the rules

(1) $\overline{\nu}(\neg\varphi) = T$ if and only if $\overline{\nu}(\varphi) = F$,
(2) $\overline{\nu}(\varphi \text{ AND } \psi) = T$ if and only if $\overline{\nu}(\varphi) = T$ and $\overline{\nu}(\psi) = T$,
(3) $\overline{\nu}(\varphi \text{ OR } \psi) = T$ if and only if $\overline{\nu}(\varphi) = T$ or $\overline{\nu}(\psi) = T$ (including the case that both are equal to T).

A set $\Sigma \subseteq \overline{S}$ is *satisfiable* if there exists a truth assignment ν such that $\overline{\nu}(\varphi) = T$ for all $\varphi \in \Sigma$. Σ is *finitely satisfiable* if every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable. Note that these concepts refer only to the internal consistency of Σ; there is so far no meaning attached to the sentence symbols themselves.

Theorem 1.6. (THE COMPACTNESS THEOREM) *A set of wffs is satisfiable if and only if it is finitely satisfiable.*

Proof. Let S be a set of sentence symbols and $\overline{S}$ the corresponding set of wffs. Assume that $\Sigma \subseteq \overline{S}$ is finitely satisfiable. Using Zorn's Lemma, let Δ be maximal

[2] Technically, $\overline{S}$ is just the absolutely free algebra generated by S with the operation symbols given in (2). We use AND and OR in place of the traditional symbols $\wedge$ and $\vee$ for conjunction and disjunction, respectively, in order to avoid confusion with the lattice operations in later chapters, while retaining the symbol $\neg$ for negation.

in $\mathfrak{P}(\overline{S})$ such that

(1) $\Sigma \subseteq \Delta$,

(2) Δ is finitely satisfiable.

We claim that for all $\varphi \in \overline{S}$, either $\varphi \in \Delta$ or $(\neg\varphi) \in \Delta$ (but of course not both).

Otherwise, by the maximality of Δ, we could find a finite subset $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \cup \{\varphi\}$ is not satisfiable, and a finite subset $\Delta_1 \subseteq \Delta$ such that $\Delta_1 \cup \{\neg\varphi\}$ is not satisfiable. But $\Delta_0 \cup \Delta_1$ is satisfiable, say by a truth assignment ν. If $\overline{\nu}(\varphi) = T$, this contradicts the choice of Δ_0, while $\overline{\nu}(\neg\varphi) = T$ contradicts the choice of Δ_1. So the claim holds.

Now define a truth assignment μ as follows. For each sentence symbol $p \in S$, define

$$\mu(p) = T \quad \text{iff} \quad p \in \Delta\ .$$

Now we claim that for all $\varphi \in \overline{S}$, $\overline{\mu}(\varphi) = T$ iff $\varphi \in \Delta$. This will yield $\overline{\mu}(\varphi) = T$ for all $\varphi \in \Sigma$, so that Σ is satisfiable.

To prove this last claim, let $G = \{\varphi \in \overline{S} : \overline{\mu}(\varphi) = T \text{ iff } \varphi \in \Delta\}$. We have $S \subseteq G$, and we need to show that G is closed under the operations $\neg$, AND and OR, so that $G = \overline{S}$.

(1) Suppose $\varphi = \neg\beta$ with $\beta \in G$. Then, using the first claim,

$$\begin{aligned} \overline{\mu}(\varphi) = T \quad &\text{iff} \quad \overline{\mu}(\beta) = F \\ &\text{iff} \quad \beta \notin \Delta \\ &\text{iff} \quad \neg\beta \in \Delta \\ &\text{iff} \quad \varphi \in \Delta\ . \end{aligned}$$

Hence $\varphi = \neg\beta \in G$.

(2) Suppose $\varphi = \alpha \text{ AND } \beta$ with $\alpha, \beta \in G$. Note that $\alpha \text{ AND } \beta \in \Delta$ iff $\alpha \in \Delta$ and $\beta \in \Delta$. For if $\alpha \text{ AND } \beta \in \Delta$, since $\{\alpha \text{ AND } \beta, \neg\alpha\}$ is not satisfiable we must have $\alpha \in \Delta$, and similarly $\beta \in \Delta$. Conversely, if $\alpha \in \Delta$ and $\beta \in \Delta$, then since $\{\alpha, \beta, \neg(\alpha \text{ AND } \beta)\}$ is not satisfiable, we have $\alpha \text{ AND } \beta \in \Delta$. Thus

$$\begin{aligned} \overline{\mu}(\varphi) = T \quad &\text{iff} \quad \overline{\mu}(\alpha) = T \text{ and } \overline{\mu}(\beta) = T \\ &\text{iff} \quad \alpha \in \Delta \text{ and } \beta \in \Delta \\ &\text{iff} \quad (\alpha \text{ AND } \beta) \in \Delta \\ &\text{iff} \quad \varphi \in \Delta. \end{aligned}$$

Hence $\varphi = (\alpha \text{ AND } \beta) \in G$.

(3) The case $\varphi = \alpha \text{ OR } \beta$ is similar to (2). □

We return to considering an infinite ordered set $\mathcal{P}$ of width k. Let $S = \{c_{xi} : x \in P,\ 1 \le i \le k\}$. We think of c_{xi} as corresponding to the statement "x is in the i-th chain." Let Σ be all sentences of the form

(a) $$c_{x1} \text{ OR } \dots \text{ OR } c_{xk}$$

for $x \in P$, and

(b) $$\neg(c_{xi} \text{ AND } c_{yi})$$

for all incomparable pairs $x, y \in P$ and $1 \leq i \leq k$. By the finite version of Dilworth's theorem, Σ is finitely satisfiable, so by the compactness theorem Σ is satisfiable, say by ν. We obtain the desired representation by putting $C_i = \{x \in P : \nu(c_{xi}) = T\}$. The sentences (a) insure that $C_1 \cup \cdots \cup C_k = P$, and the sentences (b) say that each C_i is a chain.

This completes the proof of Theorem 1.4.

A nice example due to M. Perles shows that Dilworth's theorem is no longer true when the width is allowed to be infinite [11]. Let κ be an infinite ordinal,[3] and let $\mathcal{P}$ be the direct product $\kappa \times \kappa$, ordered pointwise. Then $\mathcal{P}$ has no infinite antichains, so $w(\mathcal{P}) = \aleph_0$, but $c(\mathcal{P}) = |\kappa|$.

There is a nice discussion of the consequences and extensions of Dilworth's Theorem in Chapter 1 of [1]. Algorithmic aspects are discussed in Chapter 11 of [4], while a nice alternate proof appears in F. Galvin [5].

It is clear that the collection of all partial orders on a set X, ordered by set inclusion, is itself an ordered set $\mathcal{PO}(X)$. The least member of $\mathcal{PO}(X)$ is the equality relation, corresponding to the antichain order. The maximal members of $\mathcal{PO}(X)$ are the various total (chain) orders on X. Note that the intersection of a collection of partial orders on X is again a partial order. The next theorem, due to E. Szpilrajn, expresses an arbitrary partial ordering as an intersection of total orders [16].[4]

Theorem 1.7. *Every partial ordering on a set X is the intersection of the total orders on X containing it.*

Szpilrajn's theorem is a consequence of the next lemma.

Lemma 1.8. *Given an ordered set $(P, \leq)$ and $a \nleq b$, there exists an extension $\leq^*$ of $\leq$ such that $(P, \leq^*)$ is a chain and $b <^* a$.*

Proof. Let $a \nleq b$ in $\mathcal{P}$. Then the transitive closure of $\leq \cup (b, a)$ is a partial order extending $\leq$ in which $b <' a$. Explicitly, let

$$x \leq' y \quad \text{if} \quad \begin{cases} x \leq y \\ \text{or} \\ x \leq b \text{ and } a \leq y. \end{cases}$$

It is straightforward to check that this is a partial order.

[3]See Appendix 1.

[4]The Polish logician Edward Szpilrajn changed his last name to Marczewski in 1940 to avoid Nazi persecution, and survived the war.

If P is finite, repeated application of this construction yields a total order $\leq^*$ extending $\leq'$, so that $b <^* a$. For the infinite case, we can either use the compactness theorem, or perhaps easier Zorn's Lemma (the union of a chain of partial orders on X is again one) to obtain a total order $\leq^*$ extending $\leq'$. $\square$

Theorem 1.7 now follows, because the intersection of all such extensions contains only the pairs (c, d) with $c \leq d$.

Define the *dimension* $d(\mathcal{P})$ of an ordered set $\mathcal{P}$ to be the smallest cardinal κ such that the order $\leq$ on $\mathcal{P}$ is the intersection of κ total orders. The next result summarizes two basic facts about the dimension.

Theorem 1.9. *Let $\mathcal{P}$ be an ordered set. Then*

(1) *$d(\mathcal{P})$ is the smallest cardinal γ such that $\mathcal{P}$ can be embedded into the direct product of γ chains,*
(2) *$d(\mathcal{P}) \leq c(\mathcal{P})$.*

Proof. First suppose $\leq$ is the intersection of total orders $\leq_i$ $(i \in I)$ on P. If we let C_i be the chain $(P, \leq_i)$, then it is easy to see that the natural map $\varphi : P \to \prod_{i \in I} C_i$, with $(\varphi(x))_i = x$ for all $x \in P$, satisfies $x \leq y$ iff $\varphi(x) \leq \varphi(y)$. Hence φ is an embedding.

Conversely, assume $\varphi : P \to \prod_{i \in I} C_i$ is an embedding of P into a direct product of chains. We want to show that this leads to a representation of $\leq$ as the intersection of $|I|$ total orders. Define

$$x \; R_i \; y \quad \text{if} \quad \begin{cases} x \leq y \\ \text{or} \\ \varphi(x)_i < \varphi(y)_i \; . \end{cases}$$

You should check that R_i is a partial order extending $\leq$. By Lemma 1.8 each R_i can be extended to a total order $\leq_i$ extending $\leq$. To see that $\leq$ is the intersection of the $\leq_i$'s, suppose $x \nleq y$. Since φ is an embedding, then $\varphi(x)_i \nleq \varphi(y)_i$ for some i. Thus $\varphi(x)_i > \varphi(y)_i$, implying $y \; R_i \; x$ and hence $y \leq_i x$, or equivalently $x \nleq_i y$ (as $x \neq y$), as desired.

Thus the order on $\mathcal{P}$ is the intersection of κ total orders if and only if $\mathcal{P}$ can be embedded into the direct product of κ chains, yielding (1).

For (2), assume $P = \bigcup_{j \in J} C_j$ with each C_j a chain. Then, for each $j \in J$, the ordered set $\mathcal{O}(C_j)$ of order ideals of C_j is also a chain. Define a map $\varphi : P \to \prod_{j \in J} \mathcal{O}(C_j)$ by $(\varphi(x))_j = \{y \in C_j : y \leq x\}$. (Note $\emptyset \in \mathcal{O}(C_j)$, and $(\varphi(x))_j = \emptyset$ is certainly possible.) Then φ is clearly order-preserving. On the other hand, if $x \nleq y$ in P and $x \in C_j$, then $x \in (\varphi(x))_j$ and $x \notin (\varphi(y))_j$, so $(\varphi(x))_j \nsubseteq (\varphi(y))_j$ and $\varphi(x) \nleq \varphi(y)$. Thus P can be embedded into a direct product of $|J|$ chains. Using (1), this shows $d(P) \leq c(P)$. $\square$

Now we have three invariants defined on ordered sets: $w(P)$, $c(P)$ and $d(P)$. The exercises will provide you an opportunity to work with these in concrete cases. We have shown that $w(P) \leq c(P)$ and $d(P) \leq c(P)$, but width and dimension are independent. Indeed, if κ is an ordinal and κ^d its dual, then $\kappa \times \kappa^d$ has width $|\kappa|$ but dimension 2. It is a little harder to find examples of high dimension but low width (necessarily infinite by Dilworth's theorem), but it is possible (see [10] or [12]).

This concludes our brief introduction to ordered sets *per se*. We have covered only the most classical results of what is now an active field of research. A standard textbook is Schröder [15]; for something completely different, see Harzheim [7].

The journal **Order** is devoted to publishing results on ordered sets. The author's favorite papers in this field include Duffus and Rival [3], Jónsson and McKenzie [8], [9] and Roddy [13].

Exercises for Chapter 1

1. Draw the Hasse diagrams for all 4-element ordered sets (up to isomorphism).

2. Let N denote the positive integers. Show that the relation $a \mid b$ (a divides b) is a partial order on N. Draw the Hasse diagram for the ordered set of all divisors of 60.

3. A *partial map* on a set X is a map $\sigma : S \to X$ where $S = dom\ \sigma$ is a subset of X. Define $\sigma \leq \tau$ if $dom\ \sigma \subseteq dom\ \tau$ and $\tau(x) = \sigma(x)$ for all $x \in dom\ \sigma$. Show that the collection of all partial maps on X is an ordered set.

4. (a) Give an example of a map $f : \mathcal{P} \to \mathcal{Q}$ that is one-to-one, onto and order-preserving, but not an isomorphism.

(b) Show that the following are equivalent for ordered sets $\mathcal{P}$ and $\mathcal{Q}$.

(i) $\mathcal{P} \cong \mathcal{Q}$ (as defined before Theorem 1.1).
(ii) There exists $f : P \twoheadrightarrow Q$ such that $f(x) \leq f(y)$ iff $x \leq y$. ($\twoheadrightarrow$ means the map is onto.)
(iii) There exist $f : P \to Q$ and $g : Q \to P$, both order-preserving, with $gf = id_P$ and $fg = id_Q$.

5. Find all order ideals of the rational numbers $\mathbb{Q}$ with their usual order.

6. Prove that all chains in an ordered set $\mathcal{P}$ are finite if and only if $\mathcal{P}$ satisfies both the ACC and DCC.

7. Find $w(\mathcal{P})$, $c(\mathcal{P})$ and $d(\mathcal{P})$ for

(a) an antichain $\mathcal{A}$ with $|A| = \kappa$, where κ is a cardinal,
(b) $\mathcal{M}_\kappa$, where κ is a cardinal, the ordered set diagrammed in Figure 1.3(a).
(c) an n-crown, the ordered set diagrammed in Figure 1.3(b).
(d) $\mathfrak{P}(X)$ with X a finite set,
(e) $\mathfrak{P}(X)$ with X infinite.

8. Embed $\mathcal{M}_n$ ($2 \leq n < \infty$) into a direct product of two chains. Express the order on $\mathcal{M}_n$ as the intersection of two totally ordered extensions.

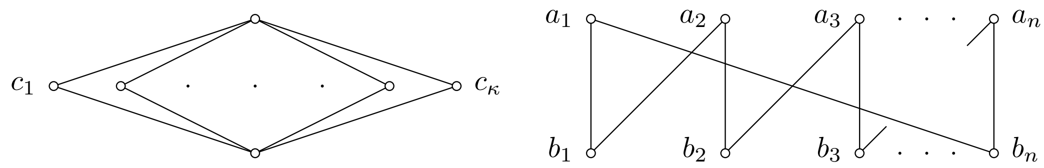

Figure 1.3

9. Let $\mathcal{P}$ be a finite ordered set with at least $ab+1$ elements. Prove that $\mathcal{P}$ contains either an antichain with $a+1$ elements, or a chain with $b+1$ elements.

10. Phillip Hall proved that if X is a finite set and $S_1, \ldots, S_n$ are subsets of X, then there is a system of distinct representatives (SDR) $a_1, \ldots, a_n$ with $a_j \in S_j$ if and only if for all $1 \leq k \leq n$ and distinct indices $i_1, \ldots, i_k$ we have $|\bigcup_{1 \leq j \leq k} S_{i_j}| \geq k$.

(a) Derive this result from Dilworth's theorem.
(b) Prove Marshall Hall's extended version: If S_i $(i \in I)$ are finite subsets of a (possibly infinite) set X, then they have an SDR if and only if the condition of P. Hall's theorem holds for every n.

11. Let R be a binary relation on a set X that contains no cycle of the form $x_0 \, R \, x_1 \, R \, \ldots R \, x_n \, R \, x_0$ with $x_i \neq x_{i+1}$. Show that the reflexive transitive closure of R is a partial order.

12. A reflexive, transitive, binary relation is called a *quasiorder.*

(a) Let R be a quasiorder on a set X. Define $x \equiv y$ if $x \, R \, y$ and $y \, R \, x$. Prove that $\equiv$ is an equivalence relation, and that R induces a partial order on $X/\equiv$.
(b) Let $\mathcal{P}$ be an ordered set, and define a relation $\ll$ on the subsets of P by $X \ll Y$ if for each $x \in X$ there exists $y \in Y$ with $x \leq y$. Verify that $\ll$ is a quasiorder.

13. Let R be any relation on a nonempty set X. Describe the smallest quasiorder containing R.

14. Let ω_1 denote the first uncountable ordinal.

(a) Let $\mathcal{P}$ be the direct product $\omega_1 \times \omega_1$. Prove that every antichain of $\mathcal{P}$ is finite, but $c(\mathcal{P}) = \aleph_1$.
(b) Let $\mathcal{Q} = \omega_1 \times \omega_1^d$. Prove that $\mathcal{Q}$ has width $\aleph_1$ but dimension 2.

15. Generalize exercise 14(a) to the direct product of two ordinals, $\mathcal{P} = \kappa \times \lambda$. Describe the maximal antichains in $\kappa \times \lambda$.

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