2. Semilattices, Lattices and Complete Lattices

There's nothing quite so fine
As an earful of Patsy Cline.
–Steve Goodman

The most important partially ordered sets come endowed with more structure than that. For example, the significant feature about $\mathcal{PO}(X)$ for Theorem 1.7 is not just its partial order, but that it is closed under arbitrary intersections. In this chapter we will meet several types of structures that arise naturally in algebra.

A *semilattice* is an algebra $\mathcal{S} = (S, *)$ satisfying, for all $x, y, z \in S$,

(1) $x * x = x$,
(2) $x * y = y * x$,
(3) $x * (y * z) = (x * y) * z$.

In other words, a semilattice is an idempotent commutative semigroup. The symbol $*$ can be replaced by any binary operation symbol, and in fact we will most often use one of $\vee$, $\wedge$, $+$ or $\cdot$, depending on the setting. The most natural example of a semilattice is $(\mathfrak{P}(X), \cap)$, or more generally any collection of subsets of X closed under intersection. For example, the semilattice $\mathcal{PO}(X)$ of partial orders on X is naturally contained in $(\mathfrak{P}(X^2), \cap)$.

Theorem 2.1. *In a semilattice $\mathcal{S}$, define $x \leq y$ if and only if $x * y = x$. Then $(S, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound. Conversely, given an ordered set $\mathcal{P}$ with that property, define $x * y = g.l.b.\,(x, y)$. Then $(P, *)$ is a semilattice.*

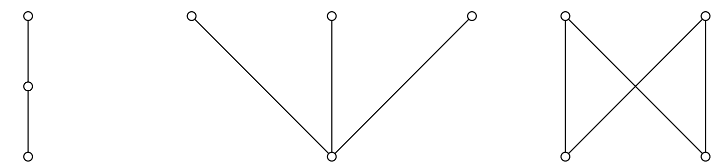
Proof. Let $(S, *)$ be a semilattice, and define $\leq$ as above. First we check that $\leq$ is a partial order.

(1) $x * x = x$ implies $x \leq x$.
(2) If $x \leq y$ and $y \leq x$, then $x = x * y = y * x = y$.
(3) If $x \leq y \leq z$, then $x * z = (x * y) * z = x * (y * z) = x * y = x$, so $x \leq z$.

Since $(x * y) * x = x * (x * y) = (x * x) * y = x * y)$ we have $x * y \leq x$; similarly $x * y \leq y$. Thus $x * y$ is a lower bound for $\{x, y\}$. To see that it is the greatest lower bound, suppose $z \leq x$ and $z \leq y$. Then $z * (x * y) = (z * x) * y = z * y = z$, so $z \leq x * y$, as desired.

The proof of the converse is likewise a direct application of the definitions, and is left to the reader. □

A semilattice with the above ordering is usually called a *meet* semilattice, and as a matter of convention $\wedge$ or $\cdot$ is used for the operation symbol. In Figure 2.1, (a) and (b) are meet semilattices, while (c) fails on several counts.



(a) (b) (c)

FIGURE 2.1

Sometimes it is more natural to use the dual order, setting $x \geq y$ iff $x * y = x$. In that case, $\mathcal{S}$ is referred to as a *join* semilattice, and the operation is denoted by $\vee$ or $+$.

A *subsemilattice* of $\mathcal{S}$ is a subset $T \subseteq S$ which is closed under the operation $*$ of $\mathcal{S}$: if $x, y \in T$ then $x * y \in T$. Of course, that makes T a semilattice in its own right, since the equations defining a semilattice still hold in $(T, *)$.[1]

Similarly, a *homomorphism* between two semilattices is a map $h : \mathcal{S} \to \mathcal{T}$ with the property that $h(x * y) = h(x) * h(y)$. An *isomorphism* is a homomorphism that is one-to-one and onto. It is worth noting that, because the operation is determined by the order and *vice versa*, two semilattices are isomorphic if and only if they are isomorphic as ordered sets.

The collection of all order ideals of a meet semilattice $\mathcal{S}$ forms a semilattice $\mathcal{O}(S)$ under set intersection. The mapping from Theorem 1.1 gives us a set representation for meet semilattices.

Theorem 2.2. *Let $\mathcal{S}$ be a meet semilattice. Define $\phi : S \to \mathcal{O}(S)$ by*

$$\phi(x) = \{y \in S : y \leq x\}.$$

Then $\mathcal{S}$ is isomorphic to $(\phi(\mathcal{S}), \cap)$.

Proof. We already know that ϕ is an order embedding of $\mathcal{S}$ into $\mathcal{O}(\mathcal{S})$. Moreover, $\phi(x \wedge y) = \phi(x) \cap \phi(y)$ because $x \wedge y$ is the greatest lower bound of x and y, so that $z \leq x \wedge y$ if and only if $z \leq x$ and $z \leq y$. □

A *lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee)$ satisfying, for all $x, y, z \in S$,

(1) $x \wedge x = x$ and $x \vee x = x$,

[1]However, it is not enough that the elements of T form a semilattice under the ordering $\leq$. For example, the sets $\{1, 2\}$, $\{1, 3\}$ and $\emptyset$ do not form a subsemilattice of $(\mathfrak{P}(\{1, 2, 3\}), \cap)$.

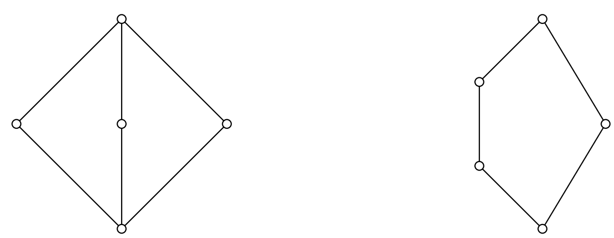
(2) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$,
(3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$,
(4) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$.

The first three pairs of axioms say that $\mathcal{L}$ is both a meet and join semilattice. The fourth pair (called the *absorption laws*) say that both operations induce the same order on L. The lattice operations are sometimes denoted by $\cdot$ and $+$; for the sake of consistency we will stick with the $\wedge$ and $\vee$ notation.

An example is the lattice $(\mathfrak{P}(X), \cap, \cup)$ of all subsets of a set X, with the usual set operations of intersection and union. This turns out not to be a very general example, because subset lattices satisfy the distributive law

$$(D) \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

The corresponding lattice equation does not hold in all lattices: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ fails, for example, in the two lattices in Figure 2.2. Hence we cannot expect to prove a representation theorem which embeds an arbitrary lattice in $(\mathfrak{P}(X), \cap, \cup)$ for some set X, although we will prove such a result for distributive lattices. A more general example would be the lattice $\mathbf{Sub}(\mathcal{G})$ of all subgroups of a group $\mathcal{G}$. Most of the remaining results in this section are designed to show how lattices arise naturally in mathematics, and to point out additional properties that some of these lattices have.



$\mathcal{M}_3$ $\qquad$ $\mathcal{N}_5$

FIGURE 2.2

Theorem 2.3. *In a lattice $\mathcal{L}$, define $x \leq y$ if and only if $x \wedge y = x$. Then $(L, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound and a least upper bound. Conversely, given an ordered set $\mathcal{P}$ with that property, define $x \wedge y = g.l.b.\,(x, y)$ and $x \vee y = l.u.b.\,(x, y)$. Then $(P, \wedge, \vee)$ is a lattice.*

The crucial observation in the proof is that, in a lattice, $x \wedge y = x$ if and only if $x \vee y = y$ by the absorption laws. The rest is a straightforward extension of Theorem 2.1.

This time we leave it up to you to figure out the correct definitions of *sublattice, homomorphism* and *isomorphism* for lattices. If a lattice has a least element, it is denoted by 0; the greatest element, if it exists, is denoted by 1. Of special importance are the *interval* (or *quotient*) sublattices, for each of which there are various notations used in the literature:

$$\begin{aligned} [b,a] = a/b &= \{x \in L : b \le x \le a\} \\ \downarrow a = a/0 = (a] &= \{x \in L : x \le a\} \\ \uparrow a = 1/a = [a) &= \{x \in L : a \le x\}. \end{aligned}$$

To avoid confusion, we will mostly use $[b,a]$ and $\downarrow a$ and $\uparrow a$.[2]

One further bit of notation will prove useful. For a subset A of an ordered set $\mathcal{P}$, let A^u denote the set of all upper bounds of A, i.e.,

$$\begin{aligned} A^u &= \{x \in P : x \ge a \text{ for all } a \in A\} \\ &= \bigcap_{a \in A} \uparrow a. \end{aligned}$$

Dually, A^ℓ is the set of all lower bounds of A,

$$\begin{aligned} A^\ell &= \{x \in P : x \le a \text{ for all } a \in A\} \\ &= \bigcap_{a \in A} \downarrow a. \end{aligned}$$

Let us consider the question of when a subset A of an ordered set $\mathcal{P}$ has a least upper bound. Clearly A^u must be nonempty, and this will certainly be the case if $\mathcal{P}$ has a greatest element. If moreover it happens that A^u has a greatest lower bound z in $\mathcal{P}$, then in fact $z \in A^u$, i.e., $a \le z$ for all $a \in A$, because each $a \in A$ is a lower bound for A^u. Therefore by definition z is the least upper bound of A. In this case we say that the join of A exists, and write $z = \bigvee A$ (treating the join as a partially defined operation).

But if $\mathcal{S}$ is a finite meet semilattice with a greatest element, then $\bigwedge A^u$ exists for every $A \subseteq S$. Thus we have the following result.

Theorem 2.4. *Let $\mathcal{S}$ be a finite meet semilattice with greatest element 1. Then $\mathcal{S}$ is a lattice with the join operation defined by*

$$x \vee y = \bigwedge \{x, y\}^u = \bigwedge (\uparrow x \cap \uparrow y).$$

This result not only yields an immediate supply of lattice examples, but it provides us with an efficient algorithm for deciding when a finite ordered set is a lattice:

[2] The notations $a/0$ and $1/a$ were historically used irrespective of whether $\mathcal{L}$ actually has a least element 0 or a greatest element 1.

if a finite ordered set $\mathcal{P}$ has a greatest element and every pair of elements has a meet, then $\mathcal{P}$ is a lattice. The dual version is of course equally useful.

Every finite subset of a lattice has a greatest lower bound and a least upper bound, but these bounds need not exist for infinite subsets. Let us define a *complete lattice* to be an ordered set $\mathcal{L}$ in which every subset A has a greatest lower bound $\bigwedge A$ and a least upper bound $\bigvee A$.[3] Clearly every finite lattice is complete, and every complete lattice is a lattice with 0 and 1 (but not conversely). Again $\mathfrak{P}(X)$ is a natural (but not very general) example of a complete lattice, and $\mathbf{Sub}(\mathcal{G})$ is a better one. The rational numbers with their natural order form a lattice that is not complete.

Likewise, a *complete meet semilattice* is an ordered set $\mathcal{S}$ with a greatest element and the property that every nonempty subset A of S has a greatest lower bound $\bigwedge A$. By convention, we define $\bigwedge \emptyset = 1$, the greatest element of $\mathcal{S}$. The analogue of Theorem 2.4 is as follows.

Theorem 2.5. *If $\mathcal{L}$ is a complete meet semilattice, then $\mathcal{L}$ is a complete lattice with the join operation defined by*

$$\bigvee A = \bigwedge A^u = \bigwedge (\bigcap_{a \in A} \uparrow a).$$

Complete lattices abound in mathematics because of their connection with closure systems. We will introduce three different ways of looking at these things, each with certain advantages, and prove that they are equivalent.

A *closure system* on a set X is a collection $\mathcal{C}$ of subsets of X that is closed under arbitrary intersections (including the empty intersection, so $\bigcap \emptyset = X \in \mathcal{C}$). The sets in $\mathcal{C}$ are called *closed* sets. By Theorem 2.5, the closed sets of a closure system form a complete lattice. Various examples come to mind:

(i) closed subsets of a topological space,
(ii) subgroups of a group,
(iii) subspaces of a vector space,
(iv) convex subsets of euclidean space $\Re^n$.
(v) order ideals of an ordered set,

You can probably think of other types of closure systems, and more will arise as we go along.

A *closure operator* on a set X is a map $\Gamma : \mathfrak{P}(X) \to \mathfrak{P}(X)$ satisfying, for all subsets $A, B \subseteq X$,

(1) $A \subseteq \Gamma(A)$,
(2) $A \subseteq B$ implies $\Gamma(A) \subseteq \Gamma(B)$,
(3) $\Gamma(\Gamma(A)) = \Gamma(A)$.

[3] We could have defined complete lattices as a type of infinitary algebra satisfying some axioms, but since these kinds of structures are not very familiar the above approach seems more natural. Following standard usage, we only allow finitary operations in an algebra (see Appendix 1). Thus a complete lattice as such, with its arbitrary operations $\bigvee A$ and $\bigwedge A$, does not count as an algebra.

The closure operators associated with the closure systems above are as follows:

(i) X a topological space and $\Gamma(A)$ the closure of A,
(ii) $\mathcal{G}$ a group and $\mathrm{Sg}(A)$ the subgroup generated by A,
(iii) $\mathcal{V}$ a vector space and $\mathrm{Span}(A)$ the set of all linear combinations of elements of A,
(iv) $\Re^n$ and $H(A)$ the convex hull of A.
(v) $\mathcal{P}$ an ordered set and $\mathcal{O}(A)$ the order ideal generated by A,

For a closure operator, a set D is called *closed* if $\Gamma(D) = D$, or equivalently (by (3)), if $D = \Gamma(A)$ for some A.

A set of *closure rules* on a set X is a collection Σ of properties $\varphi(S)$ of subsets of X, where each $\varphi(S)$ has one of the forms

$$x \in S$$

or

$$Y \subseteq S \implies z \in S$$

with x, $z \in X$ and $Y \subseteq X$. (Note that the first type of rule is a degenerate case of the second, taking $Y = \emptyset$.) A subset D of X is said to be *closed* with respect to these rules if $\varphi(D)$ is true for each $\varphi \in \Sigma$. The closure rules corresponding to our previous examples are:

(i) all rules $Y \subseteq S \implies z \in S$ where z is an accumulation point of Y,
(ii) the rule $1 \in S$ and all rules

$$x \in S \implies x^{-1} \in S$$
$$\{x, y\} \subseteq S \implies xy \in S$$

with x, $y \in G$,
(iii) $0 \in S$ and all rules $\{x, y\} \subseteq S \implies ax + by \in S$ with a, b scalars,
(iv) for all $\overline{x}$, $\overline{y} \in \Re^n$ and $0 < t < 1$, the rules $\{\overline{x}, \overline{y}\} \subseteq S \implies t\overline{x} + (1-t)\overline{y} \in S$.
(v) for all pairs with $x < y$ in $\mathcal{P}$ the rules $y \in S \implies x \in S$,

So the closure rules just list the properties that we check to determine if a set S is closed or not.

The rules in Σ can be quite arbitrary. Any subset $R \subseteq \mathfrak{P}(X) \times X$ determines a set of closure rules

$$\Sigma(R) = \{Y \subseteq S \implies z \in S : (Y, z) \in R\}.$$

Conversely, every set of rules Σ determines a subset $R(\Sigma) \subseteq \mathfrak{P}(X) \times X$. Formally, we can identify Σ and $R(\Sigma)$. Informally, it is useful to think of Σ as the set of properties $\varphi(S)$ that determine whether a subset $S \subseteq X$ is closed.

The following theorem makes explicit the connection between these ideas.

Theorem 2.6. *(1) If $\mathcal{C}$ is a closure system on a set X, then the map $\Gamma_{\mathcal{C}} : \mathfrak{P}(X) \to \mathfrak{P}(X)$ defined by*

$$\Gamma_{\mathcal{C}}(A) = \bigcap \{D \in \mathcal{C} : A \subseteq D\}$$

is a closure operator. Moreover, $\Gamma_{\mathcal{C}}(A) = A$ if and only if $A \in \mathcal{C}$.

(2) If Γ is a closure operator on a set X, let Σ_Γ be the set of all rules

$$c \in S$$

where $c \in \Gamma(\emptyset)$, and all rules

$$Y \subseteq S \implies z \in S$$

with $z \in \Gamma(Y)$. Then a set $D \subseteq X$ satisfies all the rules of Σ_Γ if and only if $\Gamma(D) = D$.

(3) If Σ is a set of closure rules on a set X, let $\mathcal{C}_\Sigma$ be the collection of all subsets of X that satisfy all the rules of Σ. Then $\mathcal{C}_\Sigma$ is a closure system.

In other words, the collection of all closed sets of a closure operator forms a complete lattice, and the property of being a closed set can be expressed in terms of rules that are clearly preserved by set intersection. It is only a slight exaggeration to say that all important lattices arise in this way. As a matter of notation, we will also use $\mathcal{C}_\Gamma$ to denote the lattice of Γ-closed sets, even though this particular variant is skipped in the statement of the theorem.

Proof. Starting with a closure system $\mathcal{C}$, define $\Gamma_{\mathcal{C}}$ as above. Observe that $\Gamma_{\mathcal{C}}(A) \in \mathcal{C}$ for any $A \subseteq X$, and $\Gamma(D) = D$ for every $D \in \mathcal{C}$. Therefore $\Gamma_{\mathcal{C}}(\Gamma_{\mathcal{C}}(A)) = \Gamma_{\mathcal{C}}(A)$, and the other axioms for a closure operator hold by elementary set theory.

Given a closure operator Γ, it is clear that $\Gamma(D) \subseteq D$ iff D satisfies all the rules of Σ_Γ. Likewise, it is immediate because of the form of the rules that $\mathcal{C}_\Sigma$ is always a closure system. □

Note that if Γ is a closure operator on a set X, then the operations on $\mathcal{C}_\Gamma$ are given by

$$\bigwedge_{i \in I} D_i = \bigcap_{i \in I} D_i$$
$$\bigvee_{i \in I} D_i = \Gamma(\bigcup_{i \in I} D_i).$$

For example, in the lattice of closed subsets of a topological space, the join is the closure of the union. In the lattice of subgroups of a group, the join of a collection of subgroups is the subgroup generated by their union. The lattice of order ideals of an ordered set is somewhat exceptional in this regard, because the union of a collection of order ideals is already an order ideal.

One type of closure operator is especially important. If $\mathcal{A} = \langle A, F, C \rangle$ is an algebra, then $S \subseteq A$ is a *subalgebra* of $\mathcal{A}$ if $c \in S$ for every constant $c \in C$, and $\{s_1, \ldots, s_n\} \subseteq S$ implies $f(s_1, \ldots, s_n) \in S$ for every basic operation $f \in F$. Of course these are closure rules, so the intersection of any collection of subalgebras of $\mathcal{A}$ is again one.[4] For a subset $B \subseteq A$, define

$$\mathrm{Sg}(B) = \bigcap \{\mathcal{S} : \mathcal{S} \text{ is a subalgebra of } \mathcal{A} \text{ and } B \subseteq S\}.$$

By Theorem 2.6, Sg is a closure operator, and $\mathrm{Sg}(B)$ is of course the subalgebra generated by B. The corresponding lattice of closed sets is $\mathcal{C}_{\mathrm{Sg}} = \mathbf{Sub}\,\mathcal{A}$, the lattice of subalgebras of $\mathcal{A}$.

Galois connections provide another source of closure operators. These are relegated to the exercises not because they are unimportant, but rather to encourage you to grapple with how they work on your own.

For completeness, we include a representation theorem.

Theorem 2.7. *If $\mathcal{L}$ is a complete lattice, define a closure operator Δ on L by*

$$\Delta(A) = \{x \in L : x \leq \bigvee A\}.$$

Then $\mathcal{L}$ is isomorphic to $\mathcal{C}_\Delta$.

The isomorphism $\varphi : \mathcal{L} \to \mathcal{C}_\Delta$ is just given by $\varphi(x) = \downarrow x$.

The representation of $\mathcal{L}$ as a closure system given by Theorem 2.7 can be greatly improved upon in some circumstances. Here we will give a better representation for lattices satisfying the ACC and DCC. In Chapter 3 we will do the same for another class called algebraic lattices.

An element q of a lattice $\mathcal{L}$ is called *join irreducible* if $q = \bigvee F$ for a finite set F implies $q \in F$, i.e., q is not the join of other elements. The set of all join irreducible elements in $\mathcal{L}$ is denoted by $J(\mathcal{L})$. Note that according to the definition, if $\mathcal{L}$ has a least element 0, then $0 \notin J(\mathcal{L})$, as $0 = \bigvee \emptyset$.[5] To include zero, let $J_0(\mathcal{L}) = J(\mathcal{L}) \cup \{0\}$.

Lemma 2.8. *If a lattice $\mathcal{L}$ satisfies the DCC, then every element of $\mathcal{L}$ is a join of finitely many join irreducible elements.*

Proof. Suppose some element of $\mathcal{L}$ is not a join of join irreducible elements. Let x be a minimal such element. Then x is not itself join irreducible, nor is it zero. So $x = \bigvee F$ for some finite set F of elements strictly below x. By the minimality

[4] If $\mathcal{A}$ has no constants, then we have to worry about the empty set. We want to allow $\emptyset$ in the subalgebra lattice in this case, but realize that it is an abuse of terminology to call it a subalgebra. The term *subuniverse* is sometimes used to avoid this problem.

[5] This convention is not universal, as *join irreducible* is sometimes defined by $q = r \vee s$ implies $q = r$ or $q = s$, which is equivalent for nonzero elements.

of x, each $f \in F$ is the join of a finite set $G_f \subseteq J(\mathcal{L})$. Then $x = \bigvee_{f \in F} \bigvee G_f$, a contradiction. $\square$

Analogously, an element q of a complete lattice $\mathcal{L}$ is said to be *completely join irreducible* if $q = \bigvee X$ implies $q \in X$ for arbitrary (possibly infinite) subsets $X \subseteq L$. If q is completely join irreducible, then q has a unique lower cover, *viz.*,

$$q_* = \bigvee \{x \in L : x < q\}.$$

Moreover, $x < q$ implies $x \leq q_*$. Let $J^*(\mathcal{L})$ denote the set of completely join irreducible elements. In general, $J^*(\mathcal{L}) \subseteq J(\mathcal{L})$, but for lattices satisfying the ACC, equality holds.

Let us consider lattices that satisfy both the ACC and DCC. By Exercise 6 of Chapter 1, these are lattices in which every chain is finite, and thus just a slight generalization of finite lattices. The representation of lattices satisfying both chain conditions as a closure system is quite straightforward.

Theorem 2.9. *Let $\mathcal{L}$ be a lattice satisfying the ACC and DCC. Let Σ be the set of all closure rules on $J(\mathcal{L})$ of the form*

$$F \subseteq S \implies q \in S$$

where q is join irreducible, F is a finite subset of $J(\mathcal{L})$, and $q \leq \bigvee F$. (Include the degenerate cases $p \in S \implies q \in S$ for $q \leq p$ in $J(\mathcal{L})$.) Then $\mathcal{L}$ is isomorphic to the lattice $\mathcal{C}_\Sigma$ of Σ-closed sets.

Proof. Define order preserving maps $f : \mathcal{L} \to \mathcal{C}_\Sigma$ and $g : \mathcal{C}_\Sigma \to \mathcal{L}$ by

$$f(x) = \downarrow x \cap J(\mathcal{L})$$
$$g(S) = \bigvee S.$$

Now $gf(x) = x$ for all $x \in L$ by Lemma 2.8. On the other hand, $fg(S) = S$ for any Σ-closed set, because by the ACC we have $\bigvee S = \bigvee F$ for some finite $F \subseteq S$, which puts every join irreducible $q \leq \bigvee F$ in S by the closure rules. $\square$

A generalization of the preceding theorem is given in Exercises 15–17 of Chapter 3.

As an example of how we might apply these ideas, suppose we want to find the subalgebra lattice of a finite algebra $\mathcal{A}$. Now **Sub** $\mathcal{A}$ is finite, and every join irreducible subalgebra is of the form $\mathrm{Sg}(a)$ for some $a \in A$ (though not necessarily conversely). Thus we may determine **Sub** $\mathcal{A}$ by first finding all the 1-generated subalgebras $\mathrm{Sg}(a)$, and then computing the joins of sets of these.[6]

[6] The art of universal algebra computation lies in making the algorithms reasonably efficient. See Ralph Freese's Universal Algebra Calculator on the webpage *http://uacalc.org*. This builds on earlier versions by Freese, Emil Kiss and Matt Valeriote.

Let us look at another type of closure operator. Of course, an ordered set need not be complete. We say that a pair $(\mathcal{L}, \phi)$ is a *completion* of the ordered set $\mathcal{P}$ if $\mathcal{L}$ is a complete lattice and ϕ is an order embedding of $\mathcal{P}$ into $\mathcal{L}$. A subset Q of a complete lattice $\mathcal{L}$ is *join dense* if for every $x \in L$,

$$x = \bigvee \{q \in Q : q \leq x\}.$$

A completion $(\mathcal{L}, \phi)$ is *join dense* if $\phi(P)$ is join dense in $\mathcal{L}$, i.e., for every $x \in L$,

$$x = \bigvee \{\phi(p) : \phi(p) \leq x\}.$$

It is not hard to see that every completion of $\mathcal{P}$ contains a join dense completion. For, given a completion $(\mathcal{L}, \phi)$ of $\mathcal{P}$, let $\mathcal{L}'$ be the set of all elements of L of the form $\bigvee \{\phi(p) : p \in A\}$ for some subset $A \subseteq P$, including $\bigvee \emptyset = 0$. Then $\mathcal{L}'$ is a complete join subsemilattice of $\mathcal{L}$, and hence a complete lattice. Moreover, $\mathcal{L}'$ contains $\phi(p)$ for every $p \in P$, and $(\mathcal{L}', \phi)$ is a join dense completion of $\mathcal{P}$. Hence we may reasonably restrict our attention to join dense completions.

Our first example of a join dense completion is the lattice of order ideals $\mathcal{O}(\mathcal{P})$. Order ideals are the closed sets of the closure operator on P given by

$$O(A) = \bigcup_{a \in A} \downarrow a,$$

and the embedding ϕ is given by $\phi(p) = \downarrow p$. Note that the union of order ideals is again an order ideal, so $\mathcal{O}(\mathcal{P})$ obeys the distributive law (D).

Another example is the *MacNeille completion* $\mathcal{M}(\mathcal{P})$, a.k.a. *normal completion, completion by cuts* ['MacNeille1937']. For subsets $S, T \subseteq P$ recall that

$$S^u = \{x \in P : x \geq s \text{ for all } s \in S\}$$
$$T^\ell = \{y \in P : y \leq t \text{ for all } t \in T\}.$$

The MacNeille completion is the lattice of closed sets of the closure operator on P given by

$$M(A) = (A^u)^\ell \ ,$$

i.e., $M(A)$ is the set of all lower bounds of all upper bounds of A. Note that $M(A)$ is an order ideal of $\mathcal{P}$. Again the map $\phi(p) = \downarrow p$ embeds $\mathcal{P}$ into $\mathcal{M}(\mathcal{P})$.

Now every join dense completion preserves all existing meets in $\mathcal{P}$: if $A \subseteq P$ and A has a greatest lower bound $b = \bigwedge A$ in $\mathcal{P}$, then $\phi(b) = \bigwedge \phi(A)$ (see Exercise 11). The MacNeille completion has the nice property that it also preserves all existing joins in $\mathcal{P}$: if A has a least upper bound $c = \bigvee A$ in $\mathcal{P}$, then $\phi(c) = \downarrow c = M(A) = \bigvee \phi(A)$.

In fact, every join dense completion corresponds to a closure operator on P.

Theorem 2.10. *Let $\mathcal{P}$ be an ordered set. If Φ is a closure operator on P such that $\Phi(\{p\}) = \downarrow p$ for all $p \in P$, then $(\mathcal{C}_\Phi, \phi)$ is a join dense completion of $\mathcal{P}$, where $\phi(p) = \downarrow p$. Conversely, if $(\mathcal{L}, \phi)$ is a join dense completion of $\mathcal{P}$, then the map Φ defined by*

$$\Phi(A) = \{q \in P : \phi(q) \leq \bigvee_{a \in A} \phi(a)\}$$

is a closure operator on P, $\Phi(\{p\}) = \downarrow p$ for all $p \in P$, and $\mathcal{C}_\Phi \cong \mathcal{L}$.

Proof. For the first part, it is clear that $(\mathcal{C}_\Phi, \phi)$ is a completion of $\mathcal{P}$. It is a join dense one because every closed set must be an order ideal, and thus for every $C \in \mathcal{C}_\Phi$,

$$\begin{aligned} C &= \bigvee \{\Phi(\{p\}) : p \in C\} \\ &= \bigvee \{\downarrow p : \downarrow p \subseteq C\} \\ &= \bigvee \{\phi(p) : \phi(p) \leq C\}. \end{aligned}$$

For the converse, it is clear that Φ defined thusly satisfies $A \subseteq \Phi(A)$, and $A \subseteq B$ implies $\Phi(A) \subseteq \Phi(B)$. But we also have $\bigvee_{q \in \Phi(A)} \phi(q) = \bigvee_{a \in A} \phi(a)$, so $\Phi(\Phi(A)) = \Phi(A)$.

To see that $\mathcal{C}_\Phi \cong \mathcal{L}$, let $f : \mathcal{C}_\Phi \to \mathcal{L}$ by $f(A) = \bigvee_{a \in A} \phi(a)$, and let $g : \mathcal{L} \to \mathcal{C}_\Phi$ by $g(x) = \{p \in P : \phi(p) \leq x\}$. Then both maps are order preserving, $fg(x) = x$ for $x \in L$ by the definition of join density, and $gf(A) = \Phi(A) = A$ for $A \in \mathcal{C}_\Phi$. Hence both maps are isomorphisms. □

There is a natural order on the closure operators on a set X.

Lemma 2.11. *Let Γ and Δ be closure operators on a set X. The following are equivalent.*

(1) $\Gamma(A) \subseteq \Delta(A)$ *for all* $A \subseteq X$.
(2) $\Delta(C) = C$ *implies* $\Gamma(C) = C$ *for all* $C \subseteq X$.

Proof. If (1) holds and $\Delta(C) = C$, then $C \subseteq \Gamma(C) \subseteq \Delta(C) = C$, whence $\Gamma(C) = C$. If (2) holds, then $\Gamma(A) \subseteq \Gamma(\Delta(A)) = \Delta(A)$. □

Let $\mathrm{Cl}(X)$ be the set of all closure operators on the set X, ordered by $\Gamma \leq \Delta$ if the conditions of Lemma 2.11 hold. For any collection Γ_i $(i \in I)$ contained in $\mathrm{Cl}(X)$, the operator $\bigwedge_{i \in I} \Gamma_i$ defined by

$$(\bigwedge_{i \in I} \Gamma_i)(A) = \bigcap_{i \in I} \Gamma_i(A)$$

is easily seen to be a closure operator, and of course the greatest lower bound of $\{\Gamma_i : i \in I\}$. Hence $\mathrm{Cl}(X)$ is a complete lattice. See Exercise 13 for a nice generalization.

Let $\mathcal{K}(\mathcal{P})$ be the collection of all closure operators on $\mathcal{P}$ such that $\Gamma(\{p\}) = \downarrow p$ for all $p \in P$. This is a complete sublattice – in fact, an interval – of $\mathrm{Cl}(P)$. The least and greatest members of $\mathcal{K}(\mathcal{P})$ are the order ideal completion and the MacNeille completion, respectively.

Theorem 2.12. *$\mathcal{K}(\mathcal{P})$ is a complete lattice with least element O and greatest element M.*

Proof. The condition $\Gamma(\{p\}) = \downarrow p$ implies that $O(A) \subseteq \Gamma(A)$ for all $A \subseteq P$, which makes O the least element of $\mathcal{K}(\mathcal{P})$. On the other hand, for any $\Gamma \in \mathcal{K}(\mathcal{P})$, if $b \geq a$ for all $a \in A$, then $\downarrow b = \Gamma(\downarrow b) \supseteq \Gamma(A)$. Thus

$$\Gamma(A) \subseteq \bigcap_{b \in A^u} (\downarrow b) = (A^u)^\ell = M(A),$$

so M is its greatest element. □

The lattices $\mathcal{K}(\mathcal{P})$ have an interesting structure, which was investigated by the author and Alex Pogel in ['NationPogel1996'].

We conclude this section with a classic theorem due to B. Knaster, A. Tarski and Anne Davis (Morel) ['Davis1955'], ['Knaster1927'], ['Tarski1955'].

Theorem 2.13. *A lattice $\mathcal{L}$ is complete if and only if every order preserving map $f : \mathcal{L} \to \mathcal{L}$ has a fixed point.*

Proof. One direction is easy. Given a complete lattice $\mathcal{L}$ and an order preserving map $f : \mathcal{L} \to \mathcal{L}$, put $A = \{x \in L : f(x) \geq x\}$. Note A is nonempty as $0 \in A$. Let $a = \bigvee A$. Since $a \geq x$ for all $x \in A$, $f(a) \geq \bigvee_{x \in A} f(x) \geq \bigvee_{x \in A} x = a$. Thus $a \in A$. But then $a \leq f(a)$ implies $f(a) \leq f^2(a)$, so also $f(a) \in A$, whence $f(a) \leq a$. Therefore $f(a) = a$.

Conversely, let $\mathcal{L}$ be a lattice that is not a complete lattice.

Claim 1: *Either $\mathcal{L}$ has no 1 or there exists a chain $C \subseteq L$ that satisfies the ACC and has no meet.* For suppose $\mathcal{L}$ has a 1 and that every chain C in $\mathcal{L}$ satisfying the ACC has a meet. We will show that every subset $S \subseteq L$ has a join, which makes $\mathcal{L}$ a complete lattice by the dual of Theorem 2.5.

Consider S^u, the set of all upper bounds of S. Note $S^u \neq \emptyset$ because $1 \in L$. Let $\mathcal{P}$ denote the collection of all chains $C \subseteq S^u$ satisfying the *ACC*, ordered by $C_1 \leq C_2$ if C_1 is a filter (dual ideal) of C_2.

The order on $\mathcal{P}$ insures that if C_i $(i \in I)$ is a chain of chains in $\mathcal{P}$, then $\bigcup_{i \in I} C_i \in \mathcal{P}$. Hence by Zorn's Lemma, $\mathcal{P}$ contains a maximal element C_m. By hypothesis $\bigwedge C_m$ exists in $\mathcal{L}$, say $\bigwedge C_m = a$. In fact, $a = \bigvee S$. For if $s \in S$, then $s \leq c$ for all $c \in C_m$, so $s \leq \bigwedge C_m = a$. Thus $a \in S^u$, i.e., a is an upper bound for S. If $a \not\leq t$ for some $t \in S^u$, then we would have $a > a \wedge t \in S^u$, and the chain $C_m \cup \{a \wedge t\}$ would contradict the maximality of C_m. Therefore $a = \bigwedge S^u = \bigvee S$. This proves Claim 1; Exercise 12 indicates why the argument is necessarily a bit involved.

If $\mathcal{L}$ has a 1, let C be a chain satisfying the ACC but having no meet; otherwise take $C = \emptyset$. Dualizing the preceding argument, let $\mathcal{Q}$ be the set of all chains $D \subseteq C^\ell$ satisfying the DCC, ordered by $D_1 \le D_2$ if D_1 is an ideal of D_2. Now $\mathcal{Q}$ could be empty, but only when C is not; if nonempty, $\mathcal{Q}$ has a maximal member D_m. Let $D = D_m$ if $\mathcal{Q} \neq \emptyset$, and $D = \emptyset$ otherwise.

Claim 2: *For all $x \in L$, either there exists $c \in C$ with $x \not\le c$, or there exists $d \in D$ with $x \not\ge d$.* Supposing otherwise, let $x \in L$ with $x \le c$ for all $c \in C$ and $x \ge d$ for all $d \in D$. (The assumption $x \in C^\ell$ means we are in the case $\mathcal{Q} \neq \emptyset$.) Since $x \in C^\ell$ and $\bigwedge C$ does not exist, there is a $y \in C^\ell$ such that $y \not\le x$. So $x \vee y > x \ge d$ for all $d \in D$, and the chain $D \cup \{x \vee y\}$ contradicts the maximality of $D = D_m$ in $\mathcal{Q}$.

Now define a map $f : \mathcal{L} \to \mathcal{L}$ as follows. For each $x \in L$, put

$$C(x) = \{c \in C : x \not\le c\} \ ,$$
$$D(x) = \{d \in D : x \not\ge d\} \ .$$

We have shown that one of these two sets is nonempty for each $x \in L$. If $C(x) \neq \emptyset$, let $f(x)$ be its largest element (using the ACC); otherwise let $f(x)$ be the least element of $D(x)$ (using the DCC). Now for any $x \in L$, either $x \not\le f(x)$ or $x \not\ge f(x)$, so f has no fixed point.

It remains to check that f is order preserving. Suppose $x \le y$. If $C(x) \neq \emptyset$ then $f(x) \in C$ and $f(x) \not\ge y$ (else $f(x) \ge y \ge x$); hence $C(y) \neq \emptyset$ and $f(y) \ge f(x)$. So assume $C(x) = \emptyset$, whence $f(x) \in D$. If perchance $C(y) \neq \emptyset$ then $f(y) \in C$, so $f(x) \le f(y)$. On the other hand, if $C(y) = \emptyset$ and $f(y) \in D$, then $x \not\ge f(y)$ (else $y \ge x \ge f(y)$), so again $f(x) \le f(y)$. Therefore f is order preserving. □

Standard textbooks on lattice theory include Birkhoff ['Birkhoff1940'], Blyth ['Blyth'], Crawley and Dilworth ['CrawleyDilworth1973'], Davey and Priestley ['DaveyPriestley'] and Grätzer ['Gratzer, Gratzer2011'], each with a slightly different perspective.

Exercises for Chapter 2

1. Draw the Hasse diagrams for
   (a) all 5 element (meet) semilattices,
   (b) all 6 element lattices,
   (c) the lattice of subspaces of the vector space $\Re^2$.
2. Prove that a lattice that has a 0 and satisfies the ACC is complete.
3. For the cyclic group $\mathbb{Z}_4$, give explicitly the subgroup lattice, the closure operator Sg, and the closure rules for subgroups.
4. Define a closure operator F on $\Re^n$ by the rules $\{\overline{x}, \overline{y}\} \subseteq S \implies t\overline{x} + (1-t)\overline{y} \in S$ for all $t \in \Re$. Describe $F(A)$ for an arbitrary subset $A \subseteq \Re^n$. What is the geometric interpretation of F?

5. Prove that the following are equivalent for a subset Q of a complete lattice $\mathcal{L}$.

(1) Q is join dense in $\mathcal{L}$, i.e., $x = \bigvee\{q \in Q : q \leq x\}$ for every $x \in L$.
(2) Every element of L is a join of elements in Q.
(3) If $y < x$ in $\mathcal{L}$, then there exists $q \in Q$ with $q \leq x$ but $q \nleq y$.

6. Let $\mathcal{L}$ be a complete lattice, and let X be a join-dense subset of L. Define a closure operator Γ on X by $\Gamma(S) = (\downarrow \bigvee S) \cap X$. Prove that $\mathcal{C}_\Gamma \cong \mathcal{L}$. (This generalizes Theorems 2.7 and 2.9, and anticipates Theorem 3.3.)

7. Find the completions $\mathcal{O}(\mathcal{P})$ and $\mathcal{M}(\mathcal{P})$ for the ordered sets in Figures 2.1 and 2.2.

8. Find the lattice $\mathcal{K}(\mathcal{P})$ of all join dense completions of the ordered sets in Figures 2.1 and 2.2.

9. Show that the MacNeille operator satisfies $M(A) = A$ iff $A = B^\ell$ for some $B \subseteq P$.

10. (a) Prove that if $(\mathcal{L}, \phi)$ is a join dense completion of the ordered set $\mathcal{P}$, then ϕ preserves all existing greatest lower bounds in $\mathcal{P}$.

(b) Prove that the MacNeille completion preserves all existing least upper bounds in $\mathcal{P}$.

11. Prove that if ϕ is an order embedding of $\mathcal{P}$ into a complete lattice $\mathcal{L}$, then ϕ extends to an order embedding of $\mathcal{M}(\mathcal{P})$ into $\mathcal{L}$.

12. Show that $\omega \times \omega_1$ has no cofinal chain. (A subset $C \subseteq P$ is *cofinal* if for every $x \in P$ there exists $c \in C$ with $x \leq c$.)

13. Following Morgan Ward ['Ward1942'], we can generalize the notion of a closure operator as follows. Let $\mathcal{L}$ be a complete lattice. (For the closure operators on a set X, $\mathcal{L}$ will be $\mathfrak{P}(X)$.) A *closure operator* on $\mathcal{L}$ is a function $f : L \to L$ that satisfies, for all $x, y \in L$,

(i) $x \leq f(x)$,
(ii) $x \leq y$ implies $f(x) \leq f(y)$,
(iii) $f(f(x)) = f(x)$.

(a) Prove that $\mathcal{C}_f = \{x \in L : f(x) = x\}$ is a complete meet subsemilattice of $\mathcal{L}$.

(b) For any complete meet subsemilattice $\mathcal{S}$ of $\mathcal{L}$, prove that the function $f_\mathcal{S}$ defined by $f_\mathcal{S}(x) = \bigwedge\{s \in S : s \geq x\}$ is a closure operator on $\mathcal{L}$.

14. Let A and B be sets, and $R \subseteq A \times B$ a relation. We define maps $\sigma : \mathfrak{P}(A) \to \mathfrak{P}(B)$ and $\pi : \mathfrak{P}(B) \to \mathfrak{P}(A)$ as follows. For $X \subseteq A$ and $Y \subseteq B$ let

$$\sigma(X) = \{b \in B : x\ R\ b \text{ for all } x \in X\}$$
$$\pi(Y) = \{a \in A : a\ R\ y \text{ for all } y \in Y\}.$$

Prove the following claims.

(a) $X \subseteq \pi\sigma(X)$ and $Y \subseteq \sigma\pi(Y)$ for all $X \subseteq A$, $Y \subseteq B$.
(b) $X \subseteq X'$ implies $\sigma(X) \supseteq \sigma(X')$, and $Y \subseteq Y'$ implies $\pi(Y) \supseteq \pi(Y')$.
(c) $\sigma(X) = \sigma\pi\sigma(X)$ and $\pi(Y) = \pi\sigma\pi(Y)$ for all $X \subseteq A$, $Y \subseteq B$.

(d) $\pi\sigma$ is a closure operator on A, and $\mathcal{C}_{\pi\sigma} = \{\pi(Y) : Y \subseteq B\}$. Likewise $\sigma\pi$ is a closure operator on B, and $\mathcal{C}_{\sigma\pi} = \{\sigma(X) : X \subseteq A\}$.
(e) $\mathcal{C}_{\pi\sigma}$ is dually isomorphic to $\mathcal{C}_{\sigma\pi}$.

The maps σ and π are said to establish a *Galois connection* between A and B. The most familiar example is when A is a set, B a group acting on A, and $a \, R \, b$ means b fixes a. The case where A is a field and B the group of automorphisms of A gives us the Galois Theory of equations. As another example, the MacNeille completion is $\mathcal{C}_{\pi\sigma}$ for the relation $\leq$ as a subset of $\mathcal{P} \times \mathcal{P}$.

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